Non-Commutative Extensions of the Standard Model

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Abstract

Four different extensions of the Standard Model to non-commutative space-time are considered. They all have the structure group $U_{\mathrm{Y}}(1) \bigotimes SU_{\mathrm{L}}(2) \bigotimes SU_{\mathrm{c}}(3)$ but differ through the way Yukawa interaction is implemented. Models based on non-commutative tensor products involve, in general, several inequivalent Seiberg–Witten maps of some (Higgs or fermionic) matter field. The non-minimal Non-Commutative Standard Model, advocated by the Munich Group, is reproduced at lowest order in the non-commutativity parameter by a particular model of this class. On the other hand, models based on hybrid Seiberg–Witten maps predict electromagnetic couplings of neutral particles like Z-boson, Higgs meson, or neutrino. The non-commutative contributions of the above Standard Model extensions at low energies are evaluated by integrating out all massive bosonic degrees of freedom.

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1 Introduction

In non-commutative field theories [1] the space-time coordinates x^μ are promoted to Hermitean operators $\hat{x}^\mu$ obeying the commutations relations

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \tag{1.1}$$

where $\theta^{\mu\nu}$ is a real, antisymmetric matrix that may be a constant, a function of the coordinate operators, or a function of both coordinate and momentum operators. While such a description has been proposed already in the early days of quantum field theory [2] its present revival is mainly due to the development of nonperturbative aspects of string theory [3].

Non-commutative field theories are conveniently studied by replacing the original multiplication law pointwise by a deformed one, that reduces to the commutative multiplication in the limit of vanishing $\theta^{\mu\nu}$. In the following a constant $\theta^{\mu\nu}$ will be considered, i.e. $[\hat{x}^\rho, \theta^{\mu\nu}] = 0$. Accordingly, one introduces an non-commutative derivative $\hat{\partial}_\mu$ with the properties

$$\left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = 0 \,, \qquad \left[\hat{\partial}_\mu, \hat{x}^\nu\right] = \delta^\nu_\mu \,. \tag{1.2}$$

It is convenient to replace $\hat{x}^\mu$ and $\hat{\partial}_\mu$ by commuting coordinates x^μ and derivatives ∂_μ defined by [4]

$$x^\mu \equiv \hat{x}^\mu - \frac{\mathrm{i}}{2}\theta^{\mu\nu}\hat{\partial}_\nu \,, \qquad \partial_\mu \equiv \hat{\partial}_\mu \,. \tag{1.3}$$

A function $\hat{F}(\hat{x})$ of non-commutative coordinates becomes a differential operator acting upon commuting coordinates

$$\hat{F}(\hat{x}) = \hat{F}(x) \exp\left(\frac{\mathrm{i}}{2} \overleftarrow{\partial}_\mu \, \theta^{\mu\nu} \, \overrightarrow{\partial}_\nu\right) \equiv \hat{F}(x) \star \,. \tag{1.4}$$

The operator on the right of $\hat{F}$ is known as the Moyal (star) product operator [5]. It is easy now to check that the commutation relations (1.1) and (1.2) are fulfilled. The Moyal product considerably simplifies the quantization problem and the derivation of Feynman rules for NC field theories.

If the quantization takes into account the full nonlinear dependence on $\theta^{\mu\nu}$, non-commutative field theories show unusual properties, like ultraviolet-infrared (UV / IR) mixing [6], or problems with unitarity for time-like [7, 8] and even space-like [9] non-commutativity parameters. By expanding the non-commutative field theory in $\theta^{\mu\nu}$, one obtains at each order a local field theory. If the asymptotic fields belong to the commutative region of space-time, the result is a unitary theory, free of UV /IR mixing, but non-renormalizable since $\theta^{\mu\nu}$ is dimensionful.

Representing non-commutative fields by Seiberg–Witten maps [10], a method of implementing any gauge theory with compact structure group in non-commutative space-time has been developed in refs. [11] and [12]. Remarkably enough, the resulting non-commutative gauge theory has the same one-loop anomalies as their commutative counterparts [13]. When applied to the full Standard Model of particle physics the method leads to a minimal non-commutative extension [14] with the same number of coupling constants as the original Standard Model.

However this extension is not unique. While no additional fields have to be incorporated, there remains ambiguities in the choice of kinetic terms for the gauge potentials and for the matter fields, including Higgs bosons. For instance, a non-minimal Non-Commutative Standard Model (nmNCSM) has been proposed in [14] and [15], the main difference to the minimal extension being due to the freedom of choosing the traces in the gauge kinetic term. The phenomenological implications of both minimal and non-minimal extensions have been considered in a number of works [16, 17, 18, 19, 20, 21, 22, 23].

A systematic discussion of the choice for matter kinetic terms is to our knowledge still missing (see however [15, 24]). In this paper we propose to use in the construction of the matter kinetic terms only those Seiberg–Witten maps required by the non-commutative Yukawa terms.

To start with, we review in Sect. 2 the construction via Seiberg–Witten (SW) maps [10] of the simplest non-commutative gauge theory. It consists of fermionic matter field minimally coupled to a gauge field. Beside gauge invariance the theory possesses a scaling symmetry that allows for simplifying the form of the action. This scaling symmetry turns out to be useful in checking the consistency of the various Standard Model extensions. If the vector bosons acquire a finite mass, one can integrate out these fields and obtain the low energy behaviour of the model. We conclude this section by computing the non-commutative contribution to the low-energy effective Hamiltonian. It describes four- and six-fermion processes.

In Sect. 3 we review the implementation of the Yukawa interaction by hybrid Seiberg–Witten maps [14] and present an alternative realization through the non-commutative tensor product [15]. Some issues concerning the choice of Seiberg–Witten maps for matter fields (fermionic and Higgs) are also discussed.

In Sect. 4 we construct two different extensions of the Standard Model, in which left-handed chiral fermions are represented by non-commutative tensor fields. In one model inequivalent Seiberg–Witten maps represent the Higgs field φ and its charge conjugate $\widetilde{\varphi}$. For both maps we assume weighted contributions to the extended action. We find that, at first order in $\theta^{\mu\nu}$, the nmNCSM is a special case. In a second model the Seiberg–Witten maps representing φ and $\widetilde{\varphi}$ are related by complex conjugation. As a result to the same left-handed chiral quark field one must associate a pair of non-commuting tensor fields, complex conjugate each other, having weighted contributions to the kinetic term.

For comparison, we present in Sect. 5 two models based upon hybrid Seiberg–Witten maps. However, in contrast to ref.[14], we use such maps through the whole non-commutative extension of the Standard Model action and not only in Yukawa couplings. For this reason, the first model of this section is somewhat different from the nmNCSM of refs. [14] and [25]. In particular, it predicts an electromagnetic interaction of Z-bosons [26] and of Higgs mesons [27]. In the second model the left-handed chiral fermions are represented by hybrid Seiberg–Witten maps. Such a construction has been used previously in GUT inspired models [15] and it is known to predict a coupling of the photon to neutrinos [26, 28, 29] with interesting astrophysical implications. Since a hybrid map does not possess its own gauge field, its contribution to the gauge sector may seem questionable. Based upon this observation we propose to construct the kinetic gauge term in strict accordance with the kinetic matter sector and to exclude contributions from all non-commuting gauge fields that are not associated with a specific Seiberg–Witten map for a

matter multiplet.

In Sect. 6 we integrate out all massive bosonic degrees of freedom of the Standard Model extensions in order to obtain their low-energy behaviour. We express in terms of physical fields the dominant non-commutative contribution to the effective interaction Hamiltonians for the models presented above.

Our conclusions are reported in the last section. Some considerations about effective actions are collected in the Appendix.

All models presented in the paper share some common features. On one hand, they predict processes which are absent in the Standard Model. On the other hand, they provide new contributions to Standard Model processes. In principle, using various experimental bounds and precision experiments, one should be able to distinguish between different non-commutative models, although in practice such a test is still difficult.

Like in the Standard Model there are no gauge anomalies in our models and the couplings to the non-commutative gravity are also anomaly free, at least in versions of non-commutative gravity based upon Seiberg–Witten maps.

The Higgs mechanism is not affected by the $\theta^{\mu\nu}$ expansion and the custodial $SU(2)$ symmetry plays the same role as in the commutative Standard Model. In particular, it allows for parameterizing the low-energy effective Hamiltonian in terms of the Fermi coupling constant.

We conclude this Introduction by the following remarks: While the constructions of the Seiberg–Witten maps can be in principle performed to any finite order in $\theta^{\mu\nu}$, we will restrict the explicit computations to first order. We shall deal exclusively with the (old) Standard Model in which all neutrinos are massless. There is however, in principle, no problem to incorporate massive neutrinos.

2 Review of Seiberg–Witten Maps

The Moyal product of two functions is a power series in the non-commutativity parameter $\theta^{\mu\nu}$ starting with the commutative product plus higher order terms chosen in such a way as to yield an associative product. It can be used to express infinitesimal non-commutative gauge transformations of matter and gauge fields

$$\delta\hat{\Psi} = \mathrm{i}\hat{\Lambda} \star \hat{\Psi}\ , \qquad\qquad \delta\hat{V}_\rho = \partial_\rho\hat{\Lambda} + \mathrm{i}\left[\hat{\Lambda} \stackrel{\star}{,} \hat{V}_\rho\right] \tag{2.1}$$

under usual gauge transformations

$$\delta\psi = \mathrm{i}\lambda\psi\ , \qquad\qquad \delta v_\rho = \partial_\rho\lambda - \mathrm{i}\left[v_\rho, \lambda\right] \equiv D_\rho\lambda\ . \tag{2.2}$$

Furthermore, any pair of non-commutative gauge transformations $\hat{\Lambda}$, $\hat{\Xi}$ has to satisfy the consistency condition [12]

$$\mathrm{i}\left(\delta_\lambda\hat{\Xi} - \delta_\xi\hat{\Lambda}\right) + \left[\hat{\Lambda} \stackrel{\star}{,} \hat{\Xi}\right] = \widehat{[\lambda, \xi]}\ . \tag{2.3}$$

In (2.1), (2.3) the symbol $\star$ includes also matrix multiplication. The suffixes λ, ξ mean that variations are performed with respect to two different gauge transformation and the right

hand side of (2.3) denotes a non-commutative gauge transformation of the commutator $[\lambda, \xi]$.

Commutative gauge fields and parameters are Lie algebra valued. The matter field ψ belongs to an arbitrary (unitary) representation of a compact gauge group with generators t_a satisfying

$$[t_a, t_b] = \mathrm{i} C^{abc} t_c \; . \tag{2.4}$$

Since the Moyal commutator, as introduced in (2.1) and (2.6) includes the anticommutator $\{t_a, t_b\}$, non-commutative fields and parameters do not belong in general to the Lie algebra, they are in the enveloping of the Lie algebra.

The Seiberg–Witten maps of the gauge theory are local solutions of (2.1), (2.3). This means that they can be expressed as formal series in $\theta^{\mu\nu}$ that at each order depend on commutative fields and a finite number of their derivatives. To first order in $\theta^{\mu\nu}$ one finds

$$\hat{\Lambda} = \lambda + \frac{1}{2}\theta^{\mu\nu}\left(c\partial_\mu \lambda v_\nu + c^* v_\nu \partial_\mu \lambda\right) \; , \tag{2.5}$$

$$\hat{\Psi} = \psi + \frac{1}{2}\theta^{\mu\nu}\left[v_\mu\left(-\partial_\nu + \mathrm{i} c v_\nu\right) + a v_{\mu\nu}\right]\psi \; , \tag{2.6}$$

$$\hat{V}_\rho = v_\rho + \frac{1}{2}\theta^{\mu\nu}\left\{-\frac{1}{2}\left\{v_\mu, \partial_\nu v_\rho + v_{\nu\rho}\right\} + D_\rho\left[b v_{\mu\nu} + \left(c - \frac{1}{2}\right) v_\mu v_\nu\right]\right\} \; , \tag{2.7}$$

where a can be complex and b is real. The parameter c is gauge dependent and will be chosen 1/2. In contrast to c, the parameters a and b multiply covariant quantities and may be present in the invariant action. Since they can be eliminated by a (covariant) field redefinition we call them scaling parameters.

From (2.6), (2.7) one can construct Seiberg–Witten maps for covariant derivatives and field strengths

$$\hat{\mathcal{D}}_\rho \star \hat{\Psi} = \partial_\rho \hat{\Psi} - \mathrm{i} \hat{V}_\rho \star \hat{\Psi} \; , \qquad \hat{V}_{\rho\sigma} = \partial_\rho \hat{V}_\sigma - \partial_\sigma \hat{V}_\rho - \mathrm{i}\left[\hat{V}_\rho \stackrel{\star}{,} \hat{V}_\sigma\right] \tag{2.8}$$

whose infinitesimal non-commutative transformations are

$$\delta \hat{\mathcal{D}}_\rho \star \hat{\Psi} = \mathrm{i}\hat{\Lambda} \star \left(\hat{\mathcal{D}}_\rho \star \hat{\Psi}\right) \; , \qquad \delta \hat{V}_{\rho\sigma} = \mathrm{i}\left[\hat{\Lambda} \stackrel{\star}{,} \hat{V}_{\rho\sigma}\right] \; . \tag{2.9}$$

By assuming usual boundary conditions at infinity and by using Stokes theorem one can immediately show that the action integral

$$\mathcal{S} = \int \mathrm{d}^4 x \left[\bar{\hat{\Psi}} \star \mathrm{i}\left(\hat{\not{\mathcal{D}}} \star \hat{\Psi}\right) - M \bar{\hat{\Psi}} \star \hat{\Psi} - \frac{1}{2g^2}\,\mathrm{Tr}\, \hat{V}_{\rho\sigma} \star \hat{V}^{\rho\sigma}\right] \tag{2.10}$$

is gauge invariant.

An evaluation of (2.10) up to first order in $\theta^{\mu\nu}$ leads to

$$\mathcal{S} = \mathcal{S}^0 + \Delta\mathcal{S}_g + \Delta\mathcal{S}_m \tag{2.11}$$

where $\mathcal{S}^0$ is the classical commutative action and $\Delta\mathcal{S}_g$, $\Delta\mathcal{S}_m$ are the non-commutative contributions to the gauge sector and to the matter sector, respectively. They are given

by

$$\mathcal{S}^0 = \mathcal{S}_m^0 + \mathcal{S}_g^0 = \int \mathrm{d}^4x \left[\bar{\psi}\left(\mathrm{i}\not{D} - M\right)\psi - \frac{1}{2g^2}\,\mathrm{Tr}\, v_{\rho\sigma}v^{\rho\sigma}\right] , \tag{2.12}$$

$$\Delta\mathcal{S}_g = \frac{1}{g^2}\int \mathrm{d}^4x \frac{1}{2}\theta^{\mu\nu}\,\mathrm{Tr}\, v^{\rho\sigma}\left(-2v_{\mu\rho}v_{\nu\sigma} + \frac{1}{2}v_{\rho\sigma}v_{\mu\nu}\right) , \tag{2.13}$$

$$\begin{aligned}\Delta\mathcal{S}_m = & \int \mathrm{d}^4x \frac{1}{2}\theta^{\mu\nu}\left[\bar{\psi}\mathrm{i}\gamma^\rho v_{\mu\rho}D_\nu\psi + (a - \mathrm{i}b)\left(-\mathrm{i}D_\rho\bar{\psi}\gamma^\rho - M\bar{\psi}\right)v_{\mu\nu}\psi\right. \\ & \left. + \left(a^* + \mathrm{i}b - \frac{1}{2}\right)\bar{\psi}v_{\mu\nu}\left(\mathrm{i}\not{D} - M\right)\psi\right] . \end{aligned} \tag{2.14}$$

Notice the presence of the parameters a and b only in the matter action where they multiply the equation of motion and can be scaled away by the following field redefinitions:

$$\psi \longrightarrow \psi - \frac{a}{2}\theta^{\mu\nu}v_{\mu\nu}\psi , \qquad v_\rho \longrightarrow v_\rho - \frac{b}{2}\theta^{\mu\nu}D_\rho v_{\mu\nu} . \tag{2.15}$$

The resulting non-commutative contribution is

$$\Delta\mathcal{S}_m = -\int \mathrm{d}^4x \left(\frac{\mathrm{i}}{2}\bar{\psi}\theta^{\mu\nu\rho}v_{\mu\nu}D_\rho\psi - \frac{1}{2}\theta^{\mu\nu}\bar{\psi}Mv_{\mu\nu}\psi\right) \tag{2.16}$$

where we introduced the completely antisymmetric tensor

$$\theta^{\mu\nu\rho} \equiv \frac{1}{2}\left(\theta^{\mu\nu}\gamma^\rho + \theta^{\nu\rho}\gamma^\mu + \theta^{\rho\mu}\gamma^\nu\right) . \tag{2.17}$$

The independence on scaling parameters has been verified up to second order in $\theta^{\mu\nu}$, in ref. [30].

Nontrivial physics is encoded in the (matter) currents. They are defined through the variation of the matter action with respect to the gauge fields. For the gauge field theory under consideration we have

$$-\,\delta_v\mathcal{S}_m^0 \equiv \int \mathrm{d}^4x J_a^\rho \delta A_\rho^a , \qquad v_\rho = -gA_\rho^a t_a , \qquad J_a^\rho = g\bar{\psi}\gamma^\rho t_a\psi \tag{2.18}$$

where we introduced explicitly the gauge coupling constant g. The non-commutative contribution to the matter current can be obtained from

$$\delta_v\Delta\mathcal{S}_m \equiv \int \mathrm{d}^4x \Delta J_a^\rho(v;x)\delta A_\rho^a(x) \tag{2.19}$$

where

$$\Delta J_a^\rho(v) = g\left[\bar{\psi}\theta^{\mu\nu\rho}\left(\mathrm{i}\overleftarrow{D}_\mu\, t_a D_\nu + \frac{1}{2}\left\{t_a, v_{\mu\nu}\right\}\right) + \theta^{\mu\rho}D_\mu\left(\bar{\psi}Mt_a\psi\right)\right] . \tag{2.20}$$

The notation emphasizes the additional dependence upon the gauge field.

If some of the gauge fields acquire mass the low energy behavior is adequately described by an effective theory obtained by integrating out the massive gauge fields. Since the gauge fields couple linearly to fermions the fist term in the effective action has the current-current form

$$\mathcal{H}_0^{\mathrm{eff}}(x) = \frac{1}{2}J_a^\mu(x)\bar{J}_{\mu a}(x) + \cdots . \tag{2.21}$$

where

$$\bar{J}_{\mu a}(x) \equiv \int \mathrm{d}^4 y D_{\mu\nu}(x-y) J^\nu_a(y) \ , \tag{2.22}$$

with $D_{\mu\nu}(x)$ the free propagator of the massive gauge field

$$D_{\mu\nu}(x) \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}kx} \frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}}{m^2 - k^2} \ , \tag{2.23}$$

and the dots stay for terms of at least cubic order in $\bar{J}_{\mu a}$.

To get a local effective theory one has to expand the propagator in inverse powers of the gauge field mass:

$$\bar{J}_{\mu a}(x) = \frac{1}{m^2} J_{\mu a}(x) + \frac{1}{m^4}\left(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box\right)\left(1 - \frac{\Box}{m^2} + \cdots\right) J^\nu_a(x) \ . \tag{2.24}$$

The effective action itself is obtained as an expansion in $1/m^2$. Actually, it makes sense to keep in this expansion only terms which do not require further regularization, such that the effective action will be expressed solely through bare parameters of the original theory.

Since the non-commutative current has additional gauge field dependence the evaluation of the low energy effective action is more involved. We assume of course, that the occurrence of a vector boson mass term in the commutative sector is the only modification suffered by (2.10). Due to the antisymmetric tensor $\theta^{\mu\nu\rho}$, the contribution induced by $\Delta\mathcal{S}_m$ remains unrenormalized and is given by

$$\begin{aligned} \Delta\mathcal{H}^m_{\mathrm{NC}} &= g\left[\mathrm{i}\partial_\mu \bar{\psi}\theta^{\mu\nu\rho} t_a \partial_\nu \psi + \theta^{\mu\rho}\partial_\mu\left(\bar{\psi} M t_a \psi\right)\right] \bar{J}_{\rho a} \\ &\quad - \frac{g^2}{2}\bar{J}_{\rho a}\left\{C^{abc}\bar{\psi}\left[\frac{\mathrm{i}}{2}\theta^{\mu\nu\rho}\left(\overleftarrow{\partial}_\mu - \overrightarrow{\partial}_\mu\right) + \theta^{\nu\rho} M\right] t_c \psi + \bar{\psi}\theta^{\mu\nu\rho}\left\{t_a, t_b\right\}\psi\partial_\mu\right\}\bar{J}_{\nu b} \\ &\quad - \mathrm{i}g^3 \bar{\psi}\theta^{\mu\nu\rho} t_a t_b t_c \psi \bar{J}_{\mu b}\bar{J}_{\nu c}\bar{J}_{\rho a} \ . \end{aligned} \tag{2.25}$$

Currents being bilinear in the fermionic fields, (2.25) describe four-, six- and eight-fermion effective interactions.

On the other hand, $\Delta\mathcal{S}_g$ contains triple and quadruple gauge field interactions, which also describe processes with more than four fermions. As shown in the Appendix, quartic gauge field interactions provide contributions of order $1/m^6$ to effective four-fermion processes depending on the regularization details. The non-commutative contribution to the effective Hamiltonian becomes thus

$$\begin{aligned} \Delta\mathcal{H}^{\mathrm{eff}}_{\mathrm{NC}} &= \frac{g^2}{m^2}\left[\mathrm{i}\partial_\mu \bar{\psi}\theta^{\mu\nu\rho} t_a \partial_\nu \psi + \theta^{\mu\rho}\partial_\mu\left(\bar{\psi} M t_a \psi\right)\right]\left[\eta_{\rho\sigma} + \frac{1}{m^2}\left(\partial_\rho\partial_\sigma - \eta_{\rho\sigma}\Box\right)\right]\bar{\psi}\gamma^\sigma t_a \psi \\ &\quad - \frac{g^4}{2m^4}\bar{\psi}\gamma_\rho t_a \psi\left\{C^{abc}\bar{\psi}\left[\frac{\mathrm{i}}{2}\theta^{\mu\nu\rho}\left(\overleftarrow{\partial}_\mu - \overrightarrow{\partial}_\mu\right) + \theta^{\nu\rho} M\right] t_c \psi \right. \\ &\quad \left. + \bar{\psi}\theta^{\mu\nu\rho}\left\{t_a, t_b\right\}\psi\partial_\mu\right\}\bar{\psi}\gamma_\nu t_b \psi + \mathcal{O}\left(\frac{1}{m^6}\right) \ . \end{aligned} \tag{2.26}$$

Notice that, in contrast with the commutative case the part of (2.26) describing six-fermion interactions is of order $1/m^4$ and does not depend upon the regularization details.

3 Yukawa Interaction and Seiberg–Witten Maps

The action of the commutative Standard Model consists of several parts representing the pure gauge (g) sector and the matter sectors (two fermionic sectors labelled by the superscript m and one Higgs sector labelled by H):

$$\mathcal{S}_{\rm SM} = \mathcal{S}^{(g)} + \mathcal{S}_{\rm matter} = \mathcal{S}^{(g)} + \left[\sum_m \mathcal{S}^{(m)} + \mathcal{S}^{(H)} \right] . \tag{3.1}$$

It is convenient to write the contribution of the matter sectors in the form

$$\begin{aligned} \mathcal{S}_{\rm matter} &= \int \mathrm{d}^4x \left\{ \sum_m \left[\bar\psi_{\rm L} \mathrm{i}\not{D}\psi_{\rm L} + \bar\chi_{\rm R}\mathrm{i}\not{D}\chi_{\rm R} + \bar\eta_{\rm R}\mathrm{i}\not{D}\eta_{\rm R} - \Big(\bar\psi_{\rm L} G^+ \varphi \chi_{\rm R} \right.\right. \\ &+ \left.\left. \bar\psi_{\rm L}\widetilde{G}^+ \widetilde{\varphi}\eta_{\rm R} + \text{h.c.}\Big) \right] + D^\rho\varphi^+ D_\rho\varphi + \frac{\lambda v^2}{4}\varphi^+\varphi - \frac{\lambda}{4}(\varphi^+\varphi)^2 \right\} \end{aligned} \tag{3.2}$$

where we omit the label m assigned to each fermionic field. Spinor and flavor indices are not shown explicitly, summation being understood. The fermions χ and η are decomposed into their left (suffix L) and right (suffix R) chiral parts, the Higgs field is denoted by φ, its charge conjugated by $\widetilde{\varphi} \equiv \mathrm{i}\tau^2\varphi$. $\psi_{\rm L}$ and φ form the doublets

$$\psi_{\rm L} = \begin{bmatrix} \eta_{\rm L} \\ \chi_{\rm L} \end{bmatrix} , \qquad \varphi = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v + H \end{bmatrix} \tag{3.3}$$

with H describing the physical Higgs meson. The gauge field associated to the matter multiplet labelled by n is

$$- v_\rho(n) \equiv g' Y_n B_\rho + g T^a_n W^a_\rho + g_s T^A_n G^A_\rho \tag{3.4}$$

where Y_n, T^a_n, T^A_n denote weak hypercharge, weak isospin and $SU(3)$ color, respectively. We shall use occasionally the abbreviations

$$v_\rho = v_\rho(\varphi) , \qquad v_{{\rm R}\rho} = v_\rho(\chi_{\rm R}) , \qquad v'_{{\rm R}\rho} = v_\rho(\eta_{\rm R}) . \tag{3.5}$$

The gauge field $v_{{\rm L}\rho}$ associated to the left chiral doublet $\psi_{\rm L}$ can be computed from

$$v_{{\rm L}\rho} = v_\rho + v_{{\rm R}\rho} = \widetilde{v}_\rho + v'_{{\rm R}\rho} \tag{3.6}$$

where $\widetilde{v}_\rho \equiv -\tau^2 v^*_\rho \tau^2$ is the gauge field associated to the charge conjugated Higgs.

The gauge fields B_ρ, W^a_ρ are expressed through physical ones by

$$Z_\rho = \cos\theta W^3_\rho - \sin\theta B_\rho , \qquad A_\rho = \sin\theta W^3_\rho + \cos\theta B_\rho , \qquad W^\pm_\rho = \frac{W^1_\rho \mp \mathrm{i} W^2_\rho}{\sqrt{2}} \tag{3.7}$$

with θ, the Weinberg angle.

The physical matter fields χ and η are obtained with help of unitary matrices $A_{\rm L,R}$, $A'_{\rm L,R}$ and chiral projection operators $P_{\rm L,R} \equiv 1/2\,(1 \mp \gamma_5)$:

$$\chi_{\rm L,R} = A_{\rm L,R} P_{\rm L,R} \chi , \qquad \eta_{\rm L,R} = A'_{\rm L,R} P_{\rm L,R} \eta , \tag{3.8}$$

such as to have diagonal mass matrices

$$M \equiv A_{\mathrm{R}}^{+} \frac{Gv}{\sqrt{2}} A_{\mathrm{L}} \;, \qquad\qquad M' \equiv A_{\mathrm{R}}^{\prime +} \frac{\widetilde{G}v}{\sqrt{2}} A_{\mathrm{L}}^{\prime} \;. \tag{3.9}$$

When both fields are massive a Cabibbo–Kobayashi–Maskawa matrix $V \equiv A_{\mathrm{L}}^{\prime +} A_{\mathrm{L}}$ can be introduced.

We will write the action for the pure Yang–Mills part of the Standard Model in the form

$$\mathcal{S}^{(g)} = -\frac{1}{2} \int \mathrm{d}^4 x \sum_n \frac{1}{h_n^2} \operatorname{Tr} v_{\rho\sigma}(n) v^{\rho\sigma}(n) \tag{3.10}$$

where certain weights, $1/h_n^2$, are assigned to the contribution of each of the gauge fields in the matter representation of the gauge group labelled by n. As we shall see in the next sections, what matter representations enter the sum may depend upon the specific non-commutative extension. The normalization conditions are

$$\frac{1}{2g'^2} = \sum_n \frac{Y_n^2}{h_n^2} \;, \qquad \frac{1}{2g^2} = \sum_n \frac{\operatorname{Tr}\left(T_n^a T_n^a\right)}{h_n^2} \;, \qquad \frac{1}{2g_s^2} = \sum_n \frac{\operatorname{Tr}\left(T_n^A T_n^A\right)}{h_n^2} \;. \tag{3.11}$$

While most of the terms in (3.2) have a straightforward non-commutative extension, for the Yukawa interaction

$$-\bar{\psi}_{\mathrm{L}} G^{+} \varphi \chi_{\mathrm{R}} + \mathrm{c.c.} \;, \tag{3.12}$$

there are several independent possibilities. The authors of ref. [14] represent the non-commutative Higgs field $\hat{\Phi}$ by a hybrid Seiberg–Witten map transforming left and right, under left- and right-handed chiral gauge groups, respectively

$$\delta\hat{\Phi} = \mathrm{i}\hat{\Lambda}_{\mathrm{L}} \star \hat{\Phi} - \mathrm{i}\hat{\Phi} \star \hat{\Lambda}_{\mathrm{R}} \;. \tag{3.13}$$

By assuming the non-commutative gauge transformations

$$\delta\hat{\Psi}_{\mathrm{L}} = \mathrm{i}\hat{\Lambda}_{\mathrm{L}} \star \hat{\Psi}_{\mathrm{L}} \;, \qquad\qquad \delta\hat{\chi}_{\mathrm{R}} = \mathrm{i}\hat{\Lambda}_{\mathrm{R}} \star \hat{\chi}_{\mathrm{R}} \tag{3.14}$$

for the maps $\hat{\Psi}_{\mathrm{L}}$ and $\hat{\chi}_{\mathrm{R}}$ associated to left-handed chiral and right-handed chiral fermions, one can check the gauge invariance of the following extension of the Yukawa interaction (3.12):

$$\mathcal{S}_{\mathrm{HSY}} = -\int \mathrm{d}^4 x \left(\bar{\hat{\Psi}}_{\mathrm{L}} G^{+} \star \hat{\Phi} \star \hat{\chi}_{\mathrm{R}} + \mathrm{h.c.} \right) \;. \tag{3.15}$$

The most general solution for the hybrid scalar Seiberg–Witten map up to first order in $\theta^{\mu\nu}$ is given by

$$\begin{aligned} \hat{\Phi} \;=\; & \phi + \frac{1}{2}\theta^{\mu\nu} \left[v_{\mathrm{L}\mu} \left(-\partial_\nu + \mathrm{i}c_{\mathrm{L}} v_{\mathrm{L}\nu} \right) \phi + \left(\partial_\mu \phi + \mathrm{i}c_{\mathrm{R}}^{*} \phi v_{\mathrm{R}\mu} \right) v_{\mathrm{R}\nu} \right. \\ & \left. - \mathrm{i} v_{\mathrm{L}\mu} \phi v_{\mathrm{R}\nu} + \alpha_{\mathrm{L}} v_{\mathrm{L}\mu\nu} \phi + \alpha_{\mathrm{R}} \phi v_{\mathrm{R}\mu\nu} \right] \end{aligned} \tag{3.16}$$

where α_{L} and α_{R} are arbitrary complex scaling parameters. The gauge dependent constants c_{L} and c_{R} do not show up in gauge invariant expressions and will be,as usual,

chosen 1/2. The contact with the commutative Higgs field φ is done by representing the commutative part ϕ as the unit matrix I_3 in color space $\phi = \varphi \bigotimes \mathrm{I}_3$.

An alternative to the use of hybrid Seiberg–Witten maps is the construction of non-commutative extensions of tensor fields and tensor product of fields. The infinitesimal gauge transformation, direct product of two commuting gauge transformations with parameters λ and λ_{R}, is represented in the non-commutative space-time by the map

$$\begin{aligned}\hat{\Omega}_{(\lambda,\lambda_{\mathrm{R}})} &= \lambda + \lambda_{\mathrm{R}} + \frac{1}{2}\theta^{\mu\nu}\left[c\partial_\mu \lambda v_\nu + c_{\mathrm{R}}\partial_\mu \lambda_{\mathrm{R}} v_{\mathrm{R}\nu} + \mathrm{h.c.}\right. \\ &\quad \left. + \left(2-\gamma_{\mathrm{R}}\right)\partial_\mu \lambda v_{\mathrm{R}\nu} + \gamma_{\mathrm{R}}\partial_\mu \lambda_{\mathrm{R}} v_\nu\right] \end{aligned} \tag{3.17}$$

with γ_{R} a new gauge dependent real constant, independent on c and c_{R}. The gauge fields associated to the two commuting groups are denoted by v_ρ and $v_{\mathrm{R}\rho}$. This formula and subsequent considerations simplify considerably in the gauge defined by $c = c_{\mathrm{R}} = 1/2$ and $\gamma_{\mathrm{R}} = 1$. We have from (3.17)

$$\hat{\Omega}_{(\lambda,\lambda_{\mathrm{R}})} = \lambda + \lambda_{\mathrm{R}} + \frac{1}{4}\theta^{\mu\nu}\left\{\partial_\mu\left(\lambda + \lambda_{\mathrm{R}}\right), v_\nu + v_{\mathrm{R}\nu}\right\} \ . \tag{3.18}$$

On the right-hand side of (3.18) one can recognize the Seiberg–Witten map $\hat{\Lambda}_{\mathrm{L}}$ of left chiral gauge transformations $\lambda_{\mathrm{L}} = \lambda_{\mathrm{R}} + \lambda$ with the corresponding gauge field $v_{\mathrm{L}\rho} = v_\rho + v_{\mathrm{R}\rho}$.

Both the non-commutative extension of tensor product $\varphi\chi_{\mathrm{R}}$

$$\begin{aligned}\widehat{\varphi\chi_{\mathrm{R}}} &= \varphi\chi_{\mathrm{R}} + \frac{1}{2}\theta^{\mu\nu}\left[v_{\mathrm{L}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v_{\mathrm{L}\nu}\right)(\varphi\chi_{\mathrm{R}}) + \mathrm{i}D_\mu\varphi D_\nu\chi_{\mathrm{R}}\right. \\ &\quad \left. + \left(\alpha v_{\mu\nu} + \alpha_{\mathrm{R}} v_{\mathrm{R}\mu\nu}\right)\varphi\chi_{\mathrm{R}}\right] \end{aligned} \tag{3.19}$$

and the Seiberg–Witten map of left-handed chiral fermion ψ_{L}

$$\hat{\Psi}_{\mathrm{L}} = \psi_{\mathrm{L}} + \frac{1}{2}\theta^{\mu\nu}\left[v_{\mathrm{L}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v_{\mathrm{L}\nu}\right) + \left(a_{\mathrm{L}} v_{\mu\nu} + a_{\mathrm{L}}^{'} v_{\mathrm{R}\mu\nu}\right)\psi_{\mathrm{L}}\right] \tag{3.20}$$

transform with the non-commutative parameter of the product gauge transformation (3.18) as

$$\delta\widehat{\varphi\chi_{\mathrm{R}}} = \mathrm{i}\hat{\Omega}_{(\lambda,\lambda_{\mathrm{R}})} \star \widehat{\varphi\chi_{\mathrm{R}}}\ , \qquad \delta\hat{\Psi}_{\mathrm{L}} = \mathrm{i}\hat{\Omega}_{(\lambda,\lambda_{\mathrm{R}})} \star \hat{\Psi}_{\mathrm{L}}\ , \tag{3.21}$$

under $\delta\varphi = \mathrm{i}\lambda\varphi$ and $\delta\chi_{\mathrm{R}} = \mathrm{i}\lambda_{\mathrm{R}}\chi_{\mathrm{R}}$.

The gauge invariant extension of the Yukawa interaction is in this case

$$\mathcal{S}_{\mathrm{TPY}} = -\int \mathrm{d}^4x \left(\bar{\hat{\Psi}}_{\mathrm{L}} G^{+} \star \widehat{\varphi\chi_{\mathrm{R}}} + \mathrm{h.c.}\right) \ . \tag{3.22}$$

A further proliferation of non-commutative standard models is caused by the different couplings of chiral fermions to φ and $\widetilde{\varphi}$. If the Seiberg-Witten maps of φ and $\widetilde{\varphi}$ are independent of each other, the non-commutative extension of the Yukawa coupling involving $\widetilde{\varphi}$ is obtained from the coupling of φ by replacing the gauge fields v_ρ, $v_{\mathrm{R}\rho}$ and the gauge parameters λ, λ_{R} through $\widetilde{v}_\rho$, $v_{\mathrm{R}\rho}^{'}$ and $\widetilde{\lambda}$, $\lambda_{\mathrm{R}}^{'}$ respectively. We shall argue in the next section that, due to (3.6), the non-commuting extensions of the direct product gauge transformations $\hat{\Omega}_{(\lambda,\lambda_{\mathrm{R}})}$ and $\hat{\Omega}_{(\widetilde{\lambda},\lambda_{\mathrm{R}}^{'})}$ can be identified.

In case that Seiberg–Witten maps of the Higgs field and its charge conjugated are related by complex conjugation, maps constructed with either the direct Moyal product, or with the opposite product

$$\circ \equiv (\star)^* = \exp\left(-\frac{\mathrm{i}}{2}\,\overleftarrow{\partial_\mu}\,\theta^{\mu\nu}\,\overrightarrow{\partial_\nu}\right)\,, \tag{3.23}$$

appear in the same non-commutative version of the Standard Model.

In the hybrid fermion model, to be discussed in Sect. 5 one starts with the full Yukawa term in the form

$$-\,\bar\chi_{\rm R} G \psi^t_{\rm L}\varphi^* - \bar\eta_{\rm R}\widetilde{G}\psi^t_{\rm L}\widetilde\varphi^* + \text{c.c.} \tag{3.24}$$

where the superscript t means transposition in the weak hypercharge-isospin space. While the Seiberg–Witten map for φ^* can be still constructed with the usual star product by the choice $\widehat{\Phi^*} \equiv -\mathrm{i}\tau^2\hat\Phi_2$, with $\hat\Phi_2$ representing $\widetilde\varphi$ in non-commutative space-time, the map of $\widetilde\varphi^*$ must be complex conjugate to $\hat\Phi_2$. As a consequence, Seiberg–Witten maps of $\chi_{\rm R}$ and $\eta_{\rm R}$ have to be constructed with opposite products. Also the left chiral fermion will be represented by two maps $\hat\Psi^t_{1\rm L}$ and $\hat\Psi^t_{2\rm L}$, opposite each other and hybrid of either φ^* and $\chi_{\rm R}$, or $\widetilde\varphi^*$ and $\eta_{\rm R}$. The general solutions of the corresponding consistency conditions are

$$\begin{aligned}
\hat\Psi^t_{1\rm L} &= \psi^t_{\rm L} + \frac{1}{2}\theta^{\mu\nu}\Big[v_{{\rm R}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v_{{\rm R}\nu}\right)\psi^t_{\rm L} + \left(-\partial_\mu\psi^t_{\rm L} + \frac{\mathrm{i}}{2}\psi^t_{\rm L}v^*_\mu\right)v^*_\nu \\
&\quad + \mathrm{i}v_{{\rm R}\mu}\psi^t_{\rm l}v^*_\nu + a'_{1\rm L}v_{{\rm R}\mu\nu}\psi^t_{\rm L} - a_{1\rm L}\psi^t_{\rm L}v^*_{\mu\nu}\Big]\,, \\
\hat\Psi^t_{2\rm L} &= \psi^t_{\rm L} - \frac{1}{2}\theta^{\mu\nu}\Big[v'_{{\rm R}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v'_{{\rm R}\nu}\right)\psi^t_{\rm L} + \left(-\partial_\mu\psi^t_{\rm L} + \frac{\mathrm{i}}{2}\psi^t_{\rm L}\widetilde v^*_\mu\right)\widetilde v^*_\nu \\
&\quad + \mathrm{i}v'_{{\rm R}\mu}\psi^t_{\rm l}\widetilde v^*_\nu - a'_{2\rm L}v'_{{\rm R}\mu\nu}\psi^t_{\rm L} + a_{2\rm L}\psi^t_{\rm L}\widetilde v^*_{\mu\nu}\Big]\,.
\end{aligned} \tag{3.25}$$

We obtain the following non-commutative extension of the Yukawa interaction (3.24):

$$\mathcal{S}_{\rm HFY} = -\int \mathrm{d}^4x\left(\bar{\hat\chi}_{\rm R}G\star\hat\Psi^t_{1\rm L}\star\widehat{\Phi^*} + \bar{\hat\eta}_{\rm R}\widetilde G\circ\hat\Psi^t_{2\rm L}\circ\hat\Phi^*_2 + \text{h.c.}\right)\,. \tag{3.26}$$

Seiberg–Witten maps constructed with opposite products can be used also in models based upon tensor product representations. The non-commutative extension of the infinitesimal parameter $\hat\Omega_{(\widetilde\lambda,\lambda'_{\rm R})}$ using the opposite Moyal product (3.23) is (in an appropriate gauge)

$$\hat\Omega_{(\widetilde\lambda,\lambda'_{\rm R})} = \widetilde\lambda + \lambda'_{\rm R} - \frac{1}{4}\theta^{\mu\nu}\left\{\partial_\mu\left(\widetilde\lambda + \lambda'_{\rm R}\right), \widetilde v_\nu + v'_{{\rm R}\nu}\right\}\,. \tag{3.27}$$

Beside the Seiberg–Witten maps $\hat\Psi_{\rm L}$, $\widehat{\varphi\chi_{\rm R}}$ transforming under (3.18) and using the usual star product, one has to introduce a second map $\hat\Psi'_{\rm L}$ for $\psi_{\rm L}$, as well as the non-commutative field $\widehat{\widetilde\varphi\eta_{\rm R}}$, transforming under (3.27) as

$$\delta\widehat{\varphi\eta_{\rm R}} = \mathrm{i}\hat\Omega_{(\widetilde\lambda,\lambda'_{\rm R})}\circ\widehat{\varphi\eta_{\rm R}}\,, \qquad \delta\hat\Psi'_{\rm L} = \mathrm{i}\hat\Omega_{(\widetilde\lambda,\lambda'_{\rm R})}\circ\hat\Psi'_{\rm L}\,. \tag{3.28}$$

The complete non-commutative Yukawa interaction is thus

$$\mathcal{S}_{\rm TPY} = -\int \mathrm{d}^4x\left(\bar{\hat\Psi}_{\rm L}G^+\star\widehat{\varphi\chi_{\rm R}} + \bar{\hat\Psi}'_{\rm L}\widetilde G^+\circ\widehat{\widetilde\varphi\eta_{\rm R}} + \text{h.c.}\right)\,. \tag{3.29}$$

In ref. [14] the Seiberg–Witten map of the Higgs field in Yukawa interaction is different from the map entering the non-commutative extension of the Higgs action. The gauge fields involved in both maps are different and have different non-commutative representatives. Moreover, in the minimal Non-Commutative Standard Model, the gauge kinetic term is extended to the non-commutative space-time by means of the Seiberg–Witten map

$$U_\rho \equiv g' \frac{y}{2} B_\rho + g \frac{\tau^a}{2} W_\rho^a + g_s \frac{\lambda^A}{2} G_\rho^A \qquad \text{with} \;\; y \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{3.30}$$

which is different from all previous non-commutative gauge field representatives.

In this work we will take a somewhat different point of view. We shall make the Seiberg–Witten maps entering Yukawa interaction dynamical, that is we shall use these maps to perform the extension of all the other terms of the Standard Model action. Since Yukawa interaction in non-commutative space-time may involve different Seiberg–Witten maps of equivalent matter representations, we associate certain weights to their non-commutative contribution to the extended action. Such a treatment also applies to the gauge sector and is similar to the approach advocated in the nmNCSM version.

The main ingredients to make Seiberg–Witten maps dynamical are non-commutative covariant derivatives (for matter fields) and field strengths (for gauge fields). For instance, the gauge map associated to the non-commutative tensor field $\hat{\Psi}_{\mathrm{L}}$ transforms under (3.17) according to

$$\delta \hat{V}_{\mathrm{L}\rho} = \partial_\rho \hat{\Omega}_{(\lambda,\lambda_{\mathrm{R}})} + \mathrm{i} \left[\hat{\Omega}_{(\lambda,\lambda_{\mathrm{R}})} \overset{\star}{,} \hat{V}_{\mathrm{L}\rho} \right] \, . \tag{3.31}$$

The dependence of the map $\hat{V}_{\mathrm{L}\rho}$ on the commuting gauge fields can be deduced from the corresponding consistency condition. Up to first order in $\theta^{\mu\nu}$ and in the gauge defined by (3.18) one gets

$$\begin{aligned} \hat{V}_{\mathrm{L}\rho} \;\; = \;\; & v_\rho + v_{\mathrm{R}\rho} + \frac{1}{2} \theta^{\mu\nu} \left[- \frac{1}{2} \left\{ v_\mu + v_{\mathrm{R}\mu}, \partial_\nu \left(v_\rho + v_{\mathrm{R}\rho} \right) + v_{\nu\rho} + v_{\mathrm{R}\nu\rho} \right\} \right. \\ & \left. + \, D_\rho \left(b_{\mathrm{L}} v_{\mu\nu} + + b_{\mathrm{L}}' v_{\mathrm{R}\mu\nu} \right) \right] \end{aligned} \tag{3.32}$$

with b_{L} and b_{L}' being arbitrary real scaling parameters.

Of course, similar considerations apply to Seiberg–Witten tensor maps constructed with the opposite Moyal product.

The covariant derivative of the hybrid maps can be obtained from their transformation properties. According to (3.13) the scalar hybrid map has the covariant derivative

$$\hat{\mathcal{D}}_\rho \star \hat{\Phi} = \partial_\rho \hat{\Phi} - \mathrm{i} \hat{V}_{\mathrm{L}\rho} \star \hat{\Phi} + \mathrm{i} \hat{\Phi} \star \hat{V}_{\mathrm{R}\rho} \, . \tag{3.33}$$

$\hat{V}_{\mathrm{L}\rho}$ and $\hat{V}_{\mathrm{R}\rho}$ stand here for the Seiberg–Witten maps of the gauge fields $v_{\mathrm{L}\rho}$ and $v_{\mathrm{R}\rho}$, respectively, and become dynamical together with $\hat{\Psi}_{\mathrm{L}}$ and $\hat{\chi}_{\mathrm{R}}$ defined in (3.14).

4 Models Based on Non-Commutative Tensor Products

The Tensor Product Model

The simplest non-commutative version of the Standard Model is obtained by representing the left chiral leptons and quarks by tensor Seiberg–Witten maps. The action of this model, hereafter called the direct tensor product model, is given by

$$
\begin{aligned}
\mathcal{S}_{\rm PM} \;=\; & \int \mathrm{d}^4x \Big\{ \bar{\hat{L}}_{\rm L} \star \mathrm{i}\left(\hat{\not{D}} \star \hat{L}_{\rm L}\right) + \bar{\hat{Q}}_{\rm L} \star \mathrm{i}\left(\hat{\not{D}} \star \hat{Q}_{\rm L}\right) \\
& + \bar{\hat{e}}_{\rm R} \star \mathrm{i}\left(\hat{\not{D}} \star \hat{e}_{\rm R}\right) + \bar{\hat{d}}_{\rm R} \star \mathrm{i}\left(\hat{\not{D}} \star \hat{d}_{\rm R}\right) + \bar{\hat{u}}_{\rm R} \star \mathrm{i}\left(\hat{\not{D}} \star \hat{u}_{\rm R}\right) \\
& - \left(\bar{\hat{L}}_{\rm L} G_e^+ \star \widehat{\varphi e_{\rm R}} + \bar{\hat{Q}}_{\rm L} G_d^+ \star \widehat{\varphi d_{\rm R}} + \bar{\hat{Q}}_{\rm L} G_u^+ \star \widehat{\tilde{\varphi} u_{\rm R}} + \text{h.c.}\right) \\
& + \sum_i w_i \left[\left(\hat{\mathcal{D}}^\rho \star \hat{\Phi}_i\right)^+ \star \left(\hat{\mathcal{D}}_\rho \star \hat{\Phi}_i\right) + \frac{\lambda v^2}{4} \hat{\Phi}_i^+ \star \hat{\Phi}_i - \frac{\lambda}{4} \hat{\Phi}_i^+ \star \hat{\Phi}_i \star \hat{\Phi}_i^+ \star \hat{\Phi}_i \right] \\
& - \frac{1}{2} \sum_{n \neq \varphi} \frac{1}{h_n^2} \operatorname{Tr} \hat{V}_{\rho\sigma}(n) \star \hat{V}^{\rho\sigma}(n) - \frac{1}{2h_\varphi^2} \sum_i w_i \operatorname{Tr} \hat{V}_{i\rho\sigma} \star \hat{V}_i^{\rho\sigma} \Big\} .
\end{aligned}
\tag{4.1}
$$

where w_1 and w_2 with $w_1 + w_2 = 1$ are the weights of the maps $\hat{\Phi}_1$ and $\hat{\Phi}_2$ of the Higgs field φ and its charge conjugate $\tilde{\varphi}$. The summation variable n labels the (inequivalent) matter representations of the Standard Model gauge group, i.e. $n = L, Q, e, d, u, \varphi$. The non-commutative gauge fields corresponding to the equivalent representations for the Higgs are denoted by $\hat{V}_1$ and $\hat{V}_2$. In order to agree with the normalization of the commutative Standard Model we set

$$
\frac{1}{h_1^2} = \frac{w_1}{h_\varphi^2} , \qquad \frac{1}{h_2^2} = \frac{w_2}{h_\varphi^2} \tag{4.2}
$$

for the weights of their contribution to the kinetic gauge term in the action. The gauge fields associated to left and right chiral fermions satisfy the following relations:

$$
v_\rho(L) = v_\rho(e) + v_\rho(\varphi) , \qquad v_\rho(Q) = v_\rho(d) + v_\rho(\varphi) = v_\rho(u) + v_\rho(\tilde{\varphi}) . \tag{4.3}
$$

The action for a typical matter sector of the tensor product model can be written in the form

$$
\begin{aligned}
\mathcal{S}_{\rm PM}^{(m)} \;=\; & \int \mathrm{d}^4x \Big[\bar{\hat{\Psi}}_{\rm L} \star \mathrm{i}\left(\hat{\not{D}} \star \hat{\Psi}_{\rm L}\right) + \bar{\hat{\chi}}_{\rm R} \star \mathrm{i}\left(\hat{\not{D}} \star \hat{\chi}_{\rm R}\right) + \bar{\hat{\eta}}_{\rm R} \star \mathrm{i}\left(\hat{\not{D}} \star \hat{\eta}_{\rm R}\right) \\
& - \left(\bar{\hat{\Psi}}_{\rm L} G^+ \star \widehat{\varphi \chi_{\rm R}} + \bar{\hat{\Psi}}_{\rm L} \tilde{G}^+ \star \widehat{\tilde{\varphi} \eta_{\rm R}} + \text{h.c.}\right) \Big]
\end{aligned}
\tag{4.4}
$$

where, for simplicity, we omit the label m of fermionic Seiberg–Witten maps. Since the couplings of the left chiral fermion to the Higgs and its charge conjugated are different, one is expecting two distinct non-commutative maps, $\hat{\Psi}_{1\rm L}$ and $\hat{\Psi}_{2\rm L}$. According to (3.20) they may differ only by terms proportional to the partial field strengths $v_{\mu\nu}$, $v_{{\rm R}\mu\nu}$ and $\tilde{v}_{\mu\nu}$, $v'_{{\rm R}\mu\nu}$. By taking $a'_{\rm L} = a_{\rm L}$ one can render both maps equivalent since they depend only on the sum $v_{{\rm L}\mu\nu}$ (see (3.6)). Similarly, the choice $b'_{\rm L} = b_{\rm L}$ in (3.32) leads to the

equivalence of the corresponding gauge maps $\hat{V}_{1\mathrm{L}\rho}$ and $\hat{V}_{2\mathrm{L}\rho}$. A similar treatment for the Seiberg–Witten product maps $\widehat{\varphi\chi_{\mathrm{R}}}$ and $\widehat{\widetilde{\varphi}\eta_{\mathrm{R}}}$ is not possible because the scaling parameters α and α_{R} in (3.21) are independent of each other. They can be however related to the scaling parameters of $\hat{\Phi}_1$ and $\hat{\chi}_{\mathrm{R}}$ (or $\hat{\Phi}_2$ and $\hat{\eta}_{\mathrm{R}}$). To get the precise relation we may use the equations of motion in the matter sector, as explained in Sect. 2. Since the scaling parameters appear at the first order in $\theta^{\mu\nu}$, it is sufficient to consider the equations of motion of the commutative Standard Model

$$
\begin{aligned}
\mathrm{i}\not{D}\chi_{\mathrm{R}} &= G\varphi^{+}\psi_{\mathrm{L}}\ , \qquad \mathrm{i}\not{D}\eta_{\mathrm{R}} = \widetilde{G}\widetilde{\varphi}\psi_{\mathrm{L}}\ , \qquad \mathrm{i}\not{D}\psi_{\mathrm{L}} = G^{+}\varphi\chi_{\mathrm{R}} + \widetilde{G}^{+}\widetilde{\varphi}\eta_{\mathrm{R}}\ , \\
D^{\rho}D_{\rho}\varphi &= \frac{\lambda v^2}{4}\varphi - \frac{\lambda}{2}\left(\varphi^{+}\varphi\right)\varphi + \sum_m\left[\psi_{\mathrm{L}}\bar{\chi}_{\mathrm{R}}G - \mathrm{i}\tau^2\left(\eta_{\mathrm{R}}\bar{\psi}_{\mathrm{L}}\widetilde{G}^{+}\right)^t\right]\ .
\end{aligned}
\tag{4.5}
$$

We insert now the Seiberg–Witten maps with arbitrary scaling parameters into $\sum_m \mathcal{S}^{(m)}_{\mathrm{PM}} + \mathcal{S}^{(H)}_{\mathrm{PM}}$ and compute the non-commutative contribution to first order in $\theta^{\mu\nu}$. After using (4.5), the various scaling factors have to be chosen as to cancel out, independently of w_1, w_2.

Let us give now the complete list of the maps occurring in (4.1):

$$
\begin{aligned}
\hat{\Psi}_{\mathrm{L}} &= \psi_{\mathrm{L}} + \frac{1}{2}\theta^{\mu\nu}\left[v_{\mathrm{L}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v_{\mathrm{L}\nu}\right) + a_{\mathrm{L}}v_{\mathrm{L}\mu\nu}\right]\psi_{\mathrm{L}}\ , \\
\widehat{\varphi\chi_{\mathrm{R}}} &= \varphi\chi_{\mathrm{R}} + \frac{1}{2}\theta^{\mu\nu}\Big[v_{\mathrm{L}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v_{\mathrm{L}\nu}\right)(\varphi\chi_{\mathrm{R}}) + \mathrm{i}D_\mu\varphi D_\nu\chi_{\mathrm{R}} \\
&\quad + \left(av_{\mu\nu} + a_{\mathrm{R}}v_{\mathrm{R}\mu\nu}\right)\varphi\chi_{\mathrm{R}}\Big]\ , \\
\widehat{\widetilde{\varphi}\eta_{\mathrm{R}}} &= \widetilde{\varphi}\eta_{\mathrm{R}} + \frac{1}{2}\theta^{\mu\nu}\Big[v_{\mathrm{L}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v_{\mathrm{L}\nu}\right)(\widetilde{\varphi}\eta_{\mathrm{R}}) + \mathrm{i}D_\mu\widetilde{\varphi}D_\nu\eta_{\mathrm{R}} \\
&\quad + \left(-a^{*}\widetilde{v}_{\mu\nu} + a^{'}_{\mathrm{R}}v^{'}_{\mathrm{R}\mu\nu}\right)\widetilde{\varphi}\eta_{\mathrm{R}}\Big]\ , \\
\hat{\chi}_{\mathrm{R}} &= \chi_{\mathrm{R}} + \frac{1}{2}\theta^{\mu\nu}\left[v_{\mathrm{R}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v_{\mathrm{R}\nu}\right) + a_{\mathrm{R}}v_{\mathrm{R}\mu\nu}\right]\chi_{\mathrm{R}}\ , \\
\hat{\eta}_{\mathrm{R}} &= \eta_{\mathrm{R}} + \frac{1}{2}\theta^{\mu\nu}\left[v^{'}_{\mathrm{R}\mu}\left(-\partial_\nu + \frac{\mathrm{i}}{2}v^{'}_{\mathrm{R}\nu}\right) + a^{'}_{\mathrm{R}}v^{'}_{\mathrm{R}\mu\nu}\right]\eta_{\mathrm{R}}\ , \\
\hat{\Phi}_1 &= \varphi + \frac{1}{2}\theta^{\mu\nu}\left[v_\mu\left(-\partial_\nu + \frac{\mathrm{i}}{2}v_\nu\right) + av_{\mu\nu}\right]\varphi\ , \\
\hat{\Phi}_2 &= \widetilde{\varphi} + \frac{1}{2}\theta^{\mu\nu}\left[\widetilde{v}_\mu\left(-\partial_\nu + \frac{\mathrm{i}}{2}\widetilde{v}_\nu\right) - a^{*}\widetilde{v}_{\mu\nu}\right]\widetilde{\varphi}\ , \\
\hat{V}_{1\rho} &= v_\rho + \frac{1}{2}\theta^{\mu\nu}\left(-\frac{1}{2}\left\{v_\mu, \partial_\nu v_\rho + v_{\nu\rho}\right\} + bv_{\mu\nu}\right)\ , \\
\hat{V}_{2\rho} &= \widetilde{v}_\rho + \frac{1}{2}\theta^{\mu\nu}\left(-\frac{1}{2}\left\{\widetilde{v}_\mu, \partial_\nu\widetilde{v}_\rho + \widetilde{v}_{\nu\rho}\right\} + bD_\rho\widetilde{v}_{\mu\nu}\right)\ , \\
\hat{V}_\rho(n) &= v_\rho(n) + \frac{1}{2}\theta^{\mu\nu}\left[-\frac{1}{2}\left\{v_\mu(n), \partial_\nu v_\rho(n) + v_{\nu\rho}(n)\right\} + bD_\rho v_{\mu\nu}(n)\right]\ ,
\end{aligned}
\tag{4.6}
$$

for $n = \mathrm{L}, \mathrm{R}, \mathrm{R}'$. While each matter sector (fermionic and scalar) has its own scaling factor (a_{L} and a), the parameter b appearing in the Seiberg–Witten maps of the gauge fields is universal.

Hence the tensor product model is consistent and one can eliminate by appropriate field redefinitions the scaling parameters from all Seiberg–Witten maps.

The non-commutative contribution of (4.4) can be written as

$$\Delta\mathcal{S}^{(m)}_{\mathrm{PM}} = \int \mathrm{d}^4x \Delta\mathcal{L}^{(m)}_{\mathrm{PM}} = \int \mathrm{d}^4x \left(\Delta\mathcal{L}_{\mathrm{L}} + \Delta\mathcal{L}_{\mathrm{R}} + \Delta\mathcal{L}'_{\mathrm{R}} - \Delta\mathcal{L}_{\mathrm{Y}} - \Delta\mathcal{L}'_{\mathrm{Y}}\right) \tag{4.7}$$

where

$$\begin{aligned}
\Delta\mathcal{L}_{\mathrm{L}} &\equiv -\frac{\mathrm{i}}{2}\bar{\psi}_{\mathrm{L}}\theta^{\mu\nu\rho}v_{\mathrm{L}\mu\nu}D_\rho\psi_{\mathrm{L}}\ , \\
\Delta\mathcal{L}_{\mathrm{R}} &\equiv -\frac{\mathrm{i}}{2}\bar{\chi}_{\mathrm{R}}\theta^{\mu\nu\rho}v_{\mathrm{R}\mu\nu}D_\rho\chi_{\mathrm{R}}\ , \qquad \Delta\mathcal{L}'_{\mathrm{R}} \equiv -\frac{\mathrm{i}}{2}\bar{\eta}_{\mathrm{R}}\theta^{\mu\nu\rho}v'_{\mathrm{R}\mu\nu}D_\rho\eta_{\mathrm{R}}\ , \\
\Delta\mathcal{L}_{\mathrm{Y}} &\equiv \frac{1}{2}\theta^{\mu\nu}\bar{\chi}_{\mathrm{R}}G\left(\mathrm{i}D_\mu\varphi^+D_\nu - \frac{1}{2}v_{\mathrm{R}\mu\nu}\varphi^+\right)\psi_{\mathrm{L}} + \mathrm{h.c.}\ , \\
\Delta\mathcal{L}'_{\mathrm{Y}} &\equiv \frac{1}{2}\theta^{\mu\nu}\bar{\eta}_{\mathrm{R}}\widetilde{G}\left(\mathrm{i}D_\mu\widetilde{\varphi}^+D_\nu - \frac{1}{2}v'_{\mathrm{R}\mu\nu}\widetilde{\varphi}^+\right)\psi_{\mathrm{L}} + \mathrm{h.c.}\ ,
\end{aligned} \tag{4.8}$$

with $v_{\mathrm{L}\mu\nu} = v_{\mu\nu} + v_{\mathrm{R}\mu\nu} = \widetilde{v}_{\mu\nu} + v'_{\mathrm{R}\mu\nu}$.

In a similar way one obtains the contribution to the Higgs action

$$\Delta\mathcal{S}^{(H)}_{\mathrm{PM}} = (w_1 - w_2)\int \mathrm{d}^4x \Delta\mathcal{L}_H \tag{4.9}$$

where

$$\begin{aligned}
\Delta\mathcal{L}_H \equiv \frac{1}{2}\theta^{\mu\nu}\Big(& D_\nu\varphi^+v_{\mu\rho}D^\rho\varphi + \mathrm{h.c.} - \frac{1}{2}D_\rho\varphi^+v_{\mu\nu}D^\rho\varphi \\
&- \frac{\lambda v^2}{8}\varphi^+v_{\mu\nu}\varphi - \frac{\lambda}{2}\varphi^+\varphi\mathrm{i}D_\mu\varphi^+D_\nu\varphi\Big)\ .
\end{aligned} \tag{4.10}$$

Finally, the non-commutative contribution to the gauge kinetic term is given by

$$\Delta\mathcal{S}^{(g)}_{\mathrm{PM}} = \int \mathrm{d}^4x \left[\sum_{n\neq\varphi}\frac{1}{h_n^2}\Delta\mathcal{L}_n + \frac{w_1 - w_2}{h_\varphi^2}\Delta\mathcal{L}_\varphi\right] \tag{4.11}$$

where

$$\Delta\mathcal{L}_n \equiv \frac{1}{2}\theta^{\mu\nu}\,\mathrm{Tr}\,v^{\rho\sigma}(n)\left[-2v_{\mu\rho}(n)v_{\nu\sigma}(n) + \frac{1}{2}v_{\rho\sigma}(n)v_{\mu\nu}(n)\right] \tag{4.12}$$

for $n = L, Q, e, d, u, \varphi$.

We would like to express these corrections in terms of physical fields. Since the gauge symmetry of the Standard Model is spontaneously broken to $U_{\mathrm{em}}(1)\otimes SU_{\mathrm{c}}(3)$ we shall use the covariant derivative

$$\nabla_\rho \equiv \partial_\rho + \mathrm{i}g_{\mathrm{s}}G_\rho + \mathrm{i}\boldsymbol{Q}\mathrm{e}A_\rho \tag{4.13}$$

with $G_\rho = T^AG^A_\rho$. Here $\boldsymbol{Q}$ is a charge operator in flavor space with the eigenvalues $Q_\nu = 0$, $Q_e = -1$, $Q_u = 2/3$ and $Q_d = -1/3$. The corresponding field strengths are $G_{\rho\sigma} \equiv T^AG^A_{\rho\sigma}$ and $F_{\rho\sigma}$. We will also find it convenient to include a coupling constant in the definition of the Z-field and its field strength:

$$\mathcal{Z}_\rho \equiv \frac{g}{2\cos\theta}Z_\rho\ , \qquad \mathcal{Z}_{\rho\sigma} = \partial_\rho\mathcal{Z}_\sigma - \partial_\sigma\mathcal{Z}_\rho\ . \tag{4.14}$$

Another useful abbreviation is

$$W^{\pm}_{\rho\sigma} \equiv \nabla_\rho W^{\pm}_\sigma - \nabla_\sigma W^{\pm}_\rho \tag{4.15}$$

where $\nabla_\rho W^{\pm}_\sigma = \partial_\rho W^{\pm}_\sigma \pm \mathrm{i}eA_\rho W^{\pm}_\sigma$. The non-commutative contribution to the full fermionic sector consists of a flavor changing (FC) and a flavor preserving (FP) part

$$\sum_m \Delta\mathcal{S}^{(m)} = \int \mathrm{d}^4x \left(\Delta\mathcal{L}^{\mathrm{FC}} + \mathrm{h.c.} + \Delta\mathcal{L}^{\mathrm{FP}}\right) . \tag{4.16}$$

This decomposition holds for all models under consideration, so we have omitted the subscript. Furthermore, we separate the lepton (l) from the quark (q) flavor changing contribution:

$$\Delta\mathcal{L}^{\mathrm{FC}} = \Delta\mathcal{L}^{\mathrm{l}} + \Delta\mathcal{L}^{\mathrm{q}} . \tag{4.17}$$

We obtain the following expressions:

$$\begin{aligned}
\Delta\mathcal{L}^{\mathrm{l}}_{\mathrm{PM}} &= W^-_\rho \frac{g}{\sqrt{2}}\bar{e}\theta^{\mu\nu\rho}\left(\mathrm{i}\overleftarrow{\nabla}_\mu \nabla_\nu - \mathcal{Z}_\nu \overleftarrow{\nabla}_\mu - \cos(2\theta)\nabla_\nu\mathcal{Z}_\mu + \frac{\mathrm{e}}{2}F_{\mu\nu}\right)\nu_{\mathrm{L}} \\
&\quad +(H+v)\frac{g}{\sqrt{2}}\bar{e}\frac{1}{2}\theta^{\mu\nu}\frac{M_e}{v}\left[W^-_\mu\left(\overleftarrow{\nabla}_\nu - 2\mathrm{i}\mathcal{Z}_\nu\right)\right]\nu_{\mathrm{L}} ,
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\Delta\mathcal{L}^{\mathrm{q}}_{\mathrm{PM}} &= W^-_\rho \frac{g}{\sqrt{2}}\bar{d}V^+\theta^{\mu\nu\rho}\Big[\mathrm{i}\overleftarrow{\nabla}_\mu \nabla_\nu + \left(\frac{4}{3}\sin^2\theta - 1\right)\mathcal{Z}_\nu \overleftarrow{\nabla}_\mu \\
&\quad - \left(1 - \frac{2}{3}\sin^2\theta\right)\nabla_\nu\mathcal{Z}_\mu - g_{\mathrm{s}}G_{\mu\nu} - \frac{\mathrm{e}}{6}F_{\mu\nu}\Big]P_{\mathrm{L}}u \\
&\quad +(H+v)\frac{g}{\sqrt{2}}\bar{d}\frac{1}{2}\theta^{\mu\nu}\Big\{\frac{M_d}{v}V^+P_{\mathrm{L}}W^-_\mu\left[\overleftarrow{\nabla}_\nu - 2\mathrm{i}\left(1 - \frac{2}{3}\sin^2\theta\right)\mathcal{Z}_\nu\right]W^-_\mu \\
&\quad + V^+\frac{M_u}{v}P_{\mathrm{R}}W^-_\mu\left[\nabla_\nu - 2\mathrm{i}\left(1 - \frac{1}{3}\sin^2\theta\right)\mathcal{Z}_\nu\right]W^-_\mu\Big\}u ,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\Delta\mathcal{L}^{\mathrm{FP}}_{\mathrm{PM}} &= \sum_f \bar{f}\frac{1}{2}\theta^{\mu\nu\rho}\Big\{\mathrm{i}\left[g_{\mathrm{s}}G_{\mu\nu} + \boldsymbol{Q}\mathrm{e}F_{\mu\nu} + 2\left(\boldsymbol{T_3}P_{\mathrm{L}} - \boldsymbol{Q}\sin^2\theta\right)\mathcal{Z}_{\mu\nu}\right] \\
&\quad \times\left[\nabla_\rho + 2\mathrm{i}\left(\boldsymbol{T_3}P_{\mathrm{L}} - \boldsymbol{Q}\sin^2\theta\right)\mathcal{Z}_\rho\right] - \frac{g^2}{4}\left(W^-_\mu W^+_{\nu\rho} + \mathrm{h.c.}\right)P_{\mathrm{L}}\Big\}f \\
&\quad +g^2W^+_\mu W^-_\nu \sum_f \bar{f}\frac{1}{2}\theta^{\mu\nu\rho}\boldsymbol{T_3}\Big\{\overleftarrow{\nabla}_\rho + \nabla_\rho \\
&\quad + 4\mathrm{i}\left[\boldsymbol{Q}\sin^2\theta - \boldsymbol{T_3}\left(2 + \cos(2\theta)\right)\right]\mathcal{Z}_\rho\Big\}P_{\mathrm{L}}f \\
&\quad +(H+v)\sum_f \bar{f}\frac{1}{2}\theta^{\mu\nu}\frac{\boldsymbol{M}}{v}\Big[\mathrm{i}\overleftarrow{\nabla}_\mu \nabla_\nu + 2\left(\boldsymbol{T_3}P_{\mathrm{L}} - \boldsymbol{Q}\sin^2\theta\right)\mathcal{Z}_\mu \overleftarrow{\nabla}_\nu \\
&\quad + 2\left(\boldsymbol{T_3}P_{\mathrm{R}} - \boldsymbol{Q}\sin^2\theta\right)\nabla_\nu\mathcal{Z}_\mu - g_{\mathrm{s}}G_{\mu\nu} - \boldsymbol{Q}\mathrm{e}F_{\mu\nu} + \mathrm{i}g^2\boldsymbol{T_3}W^+_\mu W^-_\nu\Big]f
\end{aligned} \tag{4.20}$$

where the covariant derivative $\overleftarrow{\nabla}$ is acting on all the factors after the coupling constant, or after the summation symbol. In (4.20) we introduced two new operators diagonal in

flavor space, the third component of the weak isospin $\boldsymbol{T_3}$ with eigenvalues $T_{3\nu} = 1/2$, $T_{3e} = -1/2$, $T_{3u} = 1/2$ and $T_{3\nu} = -1/2$, and the mass operator $\boldsymbol{M}$ with nontrivial eigenvalues given by

$$M_e = \mathrm{diag}\,(m_e, m_\mu, m_\tau) \ , \quad M_u = \mathrm{diag}\,(m_u, m_c, m_t) \ , \quad M_d = \mathrm{diag}\,(m_d, m_s, m_b) \ . \tag{4.21}$$

Since the Higgs contribution is proportional to the weight difference, tensor product models include the case when the Higgs sector does not contribute at all, to first order in $\theta^{\mu\nu}$. Furthermore, the action of nmNCSM exhibited in [25] is obtained by taking $w_1 = 1$ and $w_2 = 0$. In order to facilitate the comparison we present (4.10) in terms of physical fields:

$$\begin{aligned}
\Delta\mathcal{L}_H \;=\; & \frac{1}{2}\theta^{\mu\nu}\Bigg\{\frac{g^2}{2}W^+_\mu W^{-\rho}\Big[\mathrm{i}\partial_\nu H\partial_\rho H + (H+v)\cos^2\theta\partial_\nu H\mathcal{Z}_\rho \\
& + \frac{1}{2}(H+v)^2\Big(\mathrm{e}F_{\nu\rho} + \cos(2\theta)\mathcal{Z}_{\nu\rho} + 2\mathrm{i}(1+\cos^2\theta)\mathcal{Z}_\nu\mathcal{Z}_\rho\Big)\Big] \\
& + \frac{g^2}{4}W^+_{\mu\rho}\Big[\mathrm{i}(H+v)\Big(W^-_\nu\partial^\rho H + W^{-\rho}\partial_\nu H - \frac{1}{2}\delta^\rho_\nu W^-\cdot\partial H\Big) \\
& + (H+v)^2\Big(W^-_\nu\mathcal{Z}^\rho + W^{-\rho}\mathcal{Z}_\nu - \frac{1}{2}\delta^\rho_\nu W^-\cdot\mathcal{Z}\Big)\Big] + \mathrm{h.c.} \\
& + \mathcal{Z}_{\mu\rho}\Big[\partial_\nu H\partial^\rho H - \frac{1}{4}\delta^\rho_\nu(\partial H)^2 + (H+v)^2\Big(\mathcal{Z}_\nu\mathcal{Z}^\rho - \frac{1}{4}\delta^\rho_\nu\mathcal{Z}^2\Big) \\
& + \frac{\lambda}{32}\delta^\rho_\nu H^2(H+2v)^2\Big] + g^2 W^+\cdot W^-\Big[-(H+v)\cos^2\theta\mathcal{Z}_\mu\partial_\nu H \\
& + \frac{1}{8}(H+v)^2\left(\mathrm{e}F_{\mu\nu} + \cos(2\theta)\mathcal{Z}_{\mu\nu}\right)\Big] - \frac{\mathrm{i}g^2}{4}W^+_\mu W^-_\nu\Big[(\partial H)^2 + \frac{\mathrm{i}}{4}(H+v)^2 \\
& \times\Big((1+4\cos^2\theta)\mathcal{Z}^2 + \frac{g^2}{2}W^+W^- - \frac{\lambda}{4}H(H+2v)\Big)\Big]\Bigg\} \ .
\end{aligned} \tag{4.22}$$

The non-commutative version of the Standard Model gauge sector has been thoroughly discussed in [14, 15, 18] and the results can be taken over with minor changes (at least for $w_1 \neq w_2$).

The Twisted Product Model

The commutative action for the Higgs field is invariant under the replacement $\varphi \longleftrightarrow \widetilde{\varphi} = \mathrm{i}\tau^2\varphi^*$. By promoting this property to the non-commutative space-time, the Seiberg–Witten maps of $\widetilde{\varphi}$ and φ are related by complex conjugation and have to be constructed with opposite Moyal products. The left-chiral quark, which couples to both φ and $\widetilde{\varphi}$ will be represented by two Seiberg–Witten tensor maps $\hat{Q}_{\mathrm{L}}$ and $\hat{Q}'_{\mathrm{L}}$, complex conjugate each other. Similarly, the tensor products $d_{\mathrm{R}}\varphi$ and $u_{\mathrm{R}}\widetilde{\varphi}$ will be represented by maps constructed with opposite Moyal products. As a consequence, the non-commutative fields of u_{R}, $v_\rho(u)$ and $v_\rho(u) + \widetilde{v}_\rho$ will be constructed with the opposite product, while those of d_{R}, $v_\rho(d)$ and $v_\rho(u) + v_\rho$ will use the usual star product.

The action of this non-commutative version of the Standard Model, called here the twisted product model is given by

$$
\begin{aligned}
\mathcal{S}_{\mathrm{TM}} &= \int \mathrm{d}^4x \left\{ \bar{\hat{L}}_{\mathrm{L}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{L}_{\mathrm{L}} \right) + w \bar{\hat{Q}}_{\mathrm{L}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{Q}_{\mathrm{L}} \right) + w' \bar{\hat{Q}}_{\mathrm{L}}' \circ \mathrm{i} \left(\hat{\not{D}} \circ \hat{Q}_{\mathrm{L}}' \right) \right. \\
&+ \bar{\hat{e}}_{\mathrm{R}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{e}_{\mathrm{R}} \right) + \bar{\hat{d}}_{\mathrm{R}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{d}_{\mathrm{R}} \right) + \bar{\hat{u}}_{\mathrm{R}} \circ \mathrm{i} \left(\hat{\not{D}} \circ \hat{u}_{\mathrm{R}} \right) \\
&- \left(\bar{\hat{L}}_{\mathrm{L}} G_e^+ \star \widehat{\varphi e_{\mathrm{R}}} + \bar{\hat{Q}}_{\mathrm{L}} G_d^+ \star \widehat{\varphi d_{\mathrm{R}}} + \bar{\hat{Q}}_{\mathrm{L}}' G_u^+ \circ \widehat{\tilde{\varphi} u_{\mathrm{R}}} + \mathrm{h.c.} \right) \\
&+ \left(\hat{\mathcal{D}}^\rho \star \hat{\Phi} \right)^+ \star \left(\hat{\mathcal{D}}_\rho \star \hat{\Phi} \right) + \frac{\lambda v^2}{4} \hat{\Phi}^+ \star \hat{\Phi} - \frac{\lambda}{4} \hat{\Phi}^+ \star \hat{\Phi} \star \hat{\Phi}^+ \star \hat{\Phi} \\
&- \frac{1}{2} \sum_n{}' \frac{1}{h_n^2} \operatorname{Tr} \hat{V}_{\rho\sigma}(n) \star \hat{V}^{\rho\sigma}(n) - \frac{1}{2h_u^2} \operatorname{Tr} \hat{V}_{\rho\sigma}(u) \circ \hat{V}^{\rho\sigma}(u) \\
&\left. - \frac{1}{2h_Q^2} \operatorname{Tr} \left[w \hat{V}_{\rho\sigma}(Q) \star \hat{V}^{\rho\sigma}(Q) + w' \hat{V}_{\rho\sigma}'(Q) \circ \hat{V}'^{\rho\sigma}(Q) \right] \right\}
\end{aligned}
\tag{4.23}
$$

where w, w' (with $w + w' = 1$) are the weights of the tensor maps for the left-chiral quark field and the primed sum goes over $n = L, e, d, \varphi$. In order to check the consistency of this model we write the action of the general fermionic sector in a form similar to (4.4)

$$
\begin{aligned}
\mathcal{S}_{\mathrm{TM}}^{(m)} &= \int \mathrm{d}^4x \left[w \bar{\hat{\Psi}}_{\mathrm{L}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{\Psi}_{\mathrm{L}} \right) + w' \bar{\hat{\Psi}}_{\mathrm{L}}' \circ \mathrm{i} \left(\hat{\not{D}} \circ \hat{\Psi}_{\mathrm{L}}' \right) + \bar{\hat{\chi}}_{\mathrm{R}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{\chi}_{\mathrm{R}} \right) \right. \\
&\left. + \bar{\hat{\eta}}_{\mathrm{R}} \circ \mathrm{i} \left(\hat{\not{D}} \circ \hat{\eta}_{\mathrm{R}} \right) - \left(\bar{\hat{\Psi}}_{\mathrm{L}} G^+ \star \widehat{\varphi \chi_{\mathrm{R}}} + \bar{\hat{\Psi}}_{\mathrm{L}}' \tilde{G}^+ \circ \widehat{\tilde{\varphi} \eta_{\mathrm{R}}} + \mathrm{h.c.} \right) \right] .
\end{aligned}
\tag{4.24}
$$

By going through the same steps as before, one can establish the independent scaling parameters. The Seiberg–Witten maps using the star product can be read out from (4.6). We record here only the maps constructed with the opposite Moyal product:

$$
\begin{aligned}
\hat{\Psi}_{\mathrm{L}}' &= \psi_{\mathrm{L}} + \frac{1}{2} \theta^{\mu\nu} \left[-v_{\mathrm{L}\mu} \left(-\partial_\nu + \frac{\mathrm{i}}{2} v_{\mathrm{L}\nu} \right) + a_{\mathrm{L}} v_{\mathrm{L}\mu\nu} \right] \psi_{\mathrm{L}} , \\
\hat{\eta}_{\mathrm{R}} &= \eta_{\mathrm{R}} + \frac{1}{2} \theta^{\mu\nu} \left[-v_{\mathrm{R}\mu}' \left(-\partial_\nu + \frac{\mathrm{i}}{2} v_{\mathrm{R}\nu}' \right) + a_{\mathrm{R}}' v_{\mathrm{R}\mu\nu}' \right] \eta_{\mathrm{R}} , \\
\widehat{\tilde{\varphi} \eta_{\mathrm{R}}} &= \tilde{\varphi} \eta_{\mathrm{R}} + \frac{1}{2} \theta^{\mu\nu} \left[-v_{\mathrm{L}\mu} \left(-\partial_\nu + \frac{\mathrm{i}}{2} v_{\mathrm{L}\nu} \right) (\tilde{\varphi} \eta_{\mathrm{R}}) - \mathrm{i} D_\mu \tilde{\varphi} D_\nu \eta_{\mathrm{R}} \right. \\
&\left. + \left(-a^* \tilde{v}_{\mu\nu} + a_{\mathrm{R}}' v_{\mathrm{R}\mu\nu}' \right) \tilde{\varphi} \eta_{\mathrm{R}} \right] , \\
\hat{V}_\rho(n) &= v_\rho(n) + \frac{1}{2} \theta^{\mu\nu} \left[\frac{1}{2} \left\{ v_\mu(n), \partial_\nu v_\rho(n) + v_{\nu\rho}(n) \right\} + b D_\rho v_{\mu\nu}(n) \right]
\end{aligned}
\tag{4.25}
$$

for $n = \mathrm{L}', \mathrm{R}'$.

With the notations introduced in (4.8), (4.10) and (4.12), the non-commutative contributions to the action of the twisted product model take the following form:

$$
\begin{aligned}
\Delta \mathcal{S}_{\mathrm{TM}}^{(m)} &= \int \mathrm{d}^4x \left[(w - w') \Delta \mathcal{L}_{\mathrm{L}} + \Delta \mathcal{L}_{\mathrm{R}} - \Delta \mathcal{L}_{\mathrm{R}}' - \Delta \mathcal{L}_{\mathrm{Y}} + \Delta \mathcal{L}_{\mathrm{Y}}' \right) , \\
\Delta \mathcal{S}_{\mathrm{TM}}^{(H)} &= \int \mathrm{d}^4x \Delta \mathcal{L}_H \\
\Delta \mathcal{S}_{\mathrm{TM}}^{(g)} &= \int \mathrm{d}^4x \left[\sum_n{}' \frac{1}{h_n^2} \Delta \mathcal{L}_n + \frac{w - w'}{h_Q^2} \Delta \mathcal{L}_Q - \frac{1}{h_u^2} \Delta \mathcal{L}_u \right] .
\end{aligned}
\tag{4.26}
$$

When expressed through physical fields, the whole leptonic contribution remains unchanged with respect to the direct product model, e.g. $\Delta\mathcal{L}^{\mathrm{l}}_{\mathrm{TM}} = \Delta\mathcal{L}^{\mathrm{l}}_{\mathrm{PM}}$ as given by (4.18). The quark contribution to the flavor changing part is

$$\begin{aligned}
\Delta\mathcal{L}^{\mathrm{q}}_{\mathrm{TM}} \;=\;& W^-_\rho \frac{g}{\sqrt{2}}(w-w')\bar{d}V^+\theta^{\mu\nu\rho}\Big[\mathrm{i}\overleftarrow{\nabla}_\mu\nabla_\nu + \Big(\frac{4}{3}\sin^2\theta - 1\Big)\mathcal{Z}_\nu\overleftarrow{\nabla}_\mu \\
&- \Big(1-\frac{2}{3}\sin^2\theta\Big)\nabla_\nu\mathcal{Z}_\mu - g_{\mathrm{s}}G_{\mu\nu} - \frac{\mathrm{e}}{6}F_{\mu\nu}\Big]P_{\mathrm{L}}u \\
&+(H+v)\frac{g}{\sqrt{2}}\bar{d}\frac{1}{2}\theta^{\mu\nu}\Big\{\frac{M_d}{v}V^+P_{\mathrm{L}}W^-_\mu\Big[\overleftarrow{\nabla}_\nu - 2\mathrm{i}\Big(1-\frac{2}{3}\sin^2\theta\Big)\mathcal{Z}_\nu\Big]W^-_\mu \\
&- V^+\frac{M_u}{v}P_{\mathrm{R}}\Big[\nabla_\nu - 2\mathrm{i}\Big(1-\frac{1}{3}\sin^2\theta\Big)\mathcal{Z}_\nu\Big]W^-_\mu\Big\}u\,. \qquad (4.27)
\end{aligned}$$

The contribution of the flavor preserving part to the extended action can be also written in compact way as follows:

$$\begin{aligned}
\Delta\mathcal{L}^{\mathrm{FP}}_{\mathrm{TM}} \;=\;& \sum_f \bar{f}\theta^{\mu\nu\rho}\boldsymbol{T_3}\Big\{\mathrm{i}\Big[-\Big(g_{\mathrm{s}}G_{\mu\nu} + \boldsymbol{Q}\mathrm{e}F_{\mu\nu} - 2\boldsymbol{Q}\sin^2\theta\mathcal{Z}_{\mu\nu}\Big)(\gamma_5 + 2\boldsymbol{w}P_{\mathrm{L}}) \\
&+ 2(1-2\boldsymbol{w})\boldsymbol{T_3}P_{\mathrm{L}}\mathcal{Z}_{\mu\nu}\Big]\Big[\nabla_\rho + 2\mathrm{i}\Big(\boldsymbol{T_3}P_{\mathrm{L}} - \boldsymbol{Q}\sin^2\theta\Big)\mathcal{Z}_\rho\Big] \\
&- \frac{g^2}{4}(1-2\boldsymbol{w})\Big(W^-_\mu W^+_{\nu\rho} + \mathrm{h.c.}\Big)P_{\mathrm{L}}\Big\}f \\
&+\frac{g^2}{4}W^+_\mu W^-_\nu\sum_f \bar{f}\theta^{\mu\nu\rho}(1-2\boldsymbol{w})\Big\{\overleftarrow{\nabla}_\rho + \nabla_\rho \qquad (4.28) \\
&+ 4\mathrm{i}\Big[\boldsymbol{Q}\sin^2\theta - \boldsymbol{T_3}\left(2+\cos(2\theta)\right)\Big]\mathcal{Z}_\rho\Big\}P_{\mathrm{L}}f \\
&-(H+v)\sum_f \bar{f}\frac{1}{2}\theta^{\mu\nu}2\boldsymbol{T_3}\frac{\boldsymbol{M}}{v}\Big[\mathrm{i}\overleftarrow{\nabla}_\mu\nabla_\nu + 2\Big(\boldsymbol{T_3}P_{\mathrm{L}} - \boldsymbol{Q}\sin^2\theta\Big)\mathcal{Z}_\mu\overleftarrow{\nabla}_\nu \\
&+ 2\Big(\boldsymbol{T_3}P_{\mathrm{R}} - \boldsymbol{Q}\sin^2\theta\Big)\nabla_\nu\mathcal{Z}_\mu - g_{\mathrm{s}}G_{\mu\nu} - \boldsymbol{Q}\mathrm{e}F_{\mu\nu} - \mathrm{i}g^2\boldsymbol{T_3}W^+_\mu W^-_\nu\Big]f
\end{aligned}$$

where we introduced a weight operator $\boldsymbol{w}$ diagonal in flavor space with eigenvalues $w_\nu = 0$, $w_e = 1$, $w_u = w'$ and $w_d = w$. It is a simple exercise to check that the leptonic parts of (4.20) and (4.28) coincide.

A peculiarity of the twisted product model is that left- and right-handed chiral quarks carry different charges in non-commutative space-time and, as a consequence, the electromagnetic interactions violate parity. To see this we compute from (4.28) the electromagnetic interaction to lowest order in the coupling constant e, by assuming the quarks on their mass-shell. We get

$$\begin{aligned}
& A_\rho\frac{\mathrm{e}}{6}\theta^{\mu\nu}\Big\{\bar{d}\Big[\delta^\rho_\mu\Big(2w'\overleftarrow{\partial}_\nu M_d P_{\mathrm{R}} + \mathrm{h.c.}\Big) - \mathrm{i}\overleftarrow{\partial}_\mu\gamma^\rho\left(1-2w'P_{\mathrm{L}}\right)\partial_\nu\Big]d \\
&+ 2\bar{u}\Big[\delta^\rho_\mu\Big(2w\overleftarrow{\partial}_\nu M_u P_{\mathrm{R}} + \mathrm{h.c.}\Big) - \mathrm{i}\overleftarrow{\partial}_\mu\gamma^\rho\left(1-2wP_{\mathrm{L}}\right)\partial_\nu\Big]u\Big\} \qquad (4.29)
\end{aligned}$$

Similar observations apply to strong interactions.

5 Models Based upon Hybrid Seiberg–Witten Map

The Hybrid Scalar Model

In this model the Higgs field is represented by three scalar Seiberg–Witten maps $\hat{\Phi}_i$ with $i = e, d, u$, hybrid of the left and right chiral fermions labelled by i. The non-commutative action is

$$
\begin{aligned}
\mathcal{S}_{\mathrm{HS}} \;=\; & \int \mathrm{d}^4x \Big\{ \bar{\hat{L}}_{\mathrm{L}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{L}_{\mathrm{L}} \right) + \bar{\hat{e}}_{\mathrm{R}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{e}_{\mathrm{R}} \right) + \bar{\hat{Q}}_{\mathrm{L}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{Q}_{\mathrm{L}} \right) \\
& + \bar{\hat{d}}_{\mathrm{R}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{d}_{\mathrm{R}} \right) + \bar{\hat{u}}_{\mathrm{R}} \star \mathrm{i} \left(\hat{\not{D}} \star \hat{u}_{\mathrm{R}} \right) - \Big(\bar{\hat{L}}_{\mathrm{L}} G_e^+ \star \hat{\Phi}_e \star \hat{e}_{\mathrm{R}} \\
& + \bar{\hat{Q}}_{\mathrm{L}} G_d^+ \star \hat{\Phi}_d \star \hat{d}_{\mathrm{R}} + \bar{\hat{Q}}_{\mathrm{L}} G_u^+ \star \hat{\Phi}_u \star \hat{u}_{\mathrm{R}} + \mathrm{h.c.} \Big) \\
& + \sum_i w_i \,\mathrm{Tr} \left[\left(\hat{\mathcal{D}}^\rho \star \hat{\Phi}_i \right)^+ \star \left(\hat{\mathcal{D}}_\rho \star \hat{\Phi}_i \right) + \frac{\lambda v^2}{4} \hat{\Phi}_i^+ \star \hat{\Phi}_i - \frac{\lambda}{4} \hat{\Phi}_i^+ \star \hat{\Phi}_i \star \hat{\Phi}_i^+ \star \hat{\Phi}_i \right] \\
& - \frac{1}{2} \sum_n \frac{1}{h_n^2} \,\mathrm{Tr}\, \hat{V}_{\rho\sigma}(n) \star \hat{V}^{\rho\sigma}(n) \Big\}
\end{aligned} \tag{5.1}
$$

where w_i, with $i = e, d, u$ are the weights of the Higgs representatives, $\sum_i w_i = 1$. Since hybrid maps do not have their own gauge Seiberg–Witten maps, we *assume* that the kinetic gauge term receives contributions only from $n = L, Q, e, d, u$.

The consistency check of the model leads to a single (complex) parameter for all the three hybrid maps. Denoting by $\hat{\Phi}$, $\hat{\Phi}'$ the representatives of the Higgs field φ and its charge conjugate $\widetilde{\varphi}$, one finds (compare to (3.16))

$$
\begin{aligned}
\hat{\Phi} \;=\; & \phi + \frac{1}{2}\theta^{\mu\nu} \Big[v_{\mathrm{L}\mu} \left(-\partial_\nu + \frac{\mathrm{i}}{2} v_{\mathrm{L}\nu} \right) \phi + \phi \left(\overleftarrow{\partial}_\mu + \frac{\mathrm{i}}{2} v_{\mathrm{R}\mu} \right) v_{\mathrm{R}\nu} \\
& - \mathrm{i} v_{\mathrm{L}\mu} \phi v_{\mathrm{R}\nu} + a \left(v_{\mathrm{L}\mu\nu}\phi - \phi v_{\mathrm{R}\mu\nu} \right) \Big] \;, \\
\hat{\Phi}' \;=\; & \widetilde{\phi} + \frac{1}{2}\theta^{\mu\nu} \Big[v_{\mathrm{L}\mu} \left(-\partial_\nu + \frac{\mathrm{i}}{2} v_{\mathrm{L}\nu} \right) \widetilde{\phi} + \widetilde{\phi} \left(\overleftarrow{\partial}_\mu + \frac{\mathrm{i}}{2} v'_{\mathrm{R}\mu} \right) v'_{\mathrm{R}\nu} \\
& - \mathrm{i} v_{\mathrm{L}\mu} \widetilde{\phi} v'_{\mathrm{R}\nu} + a \left(v_{\mathrm{L}\mu\nu}\widetilde{\phi} - \widetilde{\phi} v'_{\mathrm{R}\mu\nu} \right) \Big]
\end{aligned} \tag{5.2}
$$

where $\phi = \varphi \otimes \mathrm{I}_3$ and $\widetilde{\phi} = \widetilde{\varphi} \otimes \mathrm{I}_3$. Field redefinitions can be now performed owing to the identities

$$
v_{\mathrm{L}\mu\nu}\phi = \phi v_{\mathrm{R}\mu\nu} + v_{\mu\nu}\varphi \otimes \mathrm{I}_3 \;, \qquad v_{\mathrm{L}\mu\nu}\widetilde{\phi} = \widetilde{\phi} v'_{\mathrm{R}\mu\nu} + \widetilde{v}_{\mu\nu}\widetilde{\varphi} \otimes \mathrm{I}_3 \;. \tag{5.3}
$$

The leptonic and quark sectors of the hybrid scalar and of the tensor product models coincide. The Higgs sector contributes to the extended action with the following expression:

$$
\Delta\mathcal{S}_{\mathrm{HS}}^{(H)} = \int \mathrm{d}^4x \left[(w_e + w_d - w_u)\, \Delta\mathcal{L}_H + \left(\frac{2w_u - w_d}{3} - w_e \right) \mathcal{E}_{\mathrm{HS}} \right] \tag{5.4}
$$

where $\Delta\mathcal{L}_H$ is given by (4.10) and

$$
\mathcal{E}_{\mathrm{HS}} \equiv \frac{1}{2}\theta^{\mu\nu} \left\{ 2D^\rho\varphi^+ D_\mu\varphi + \mathrm{h.c.} + \delta_\mu^\rho \left[-D\varphi^+ \cdot D\varphi + \frac{\lambda}{4}\varphi^+\varphi \left(\varphi^+\varphi - v^2 \right) \right] \right\} g' B_{\nu\rho} \;. \tag{5.5}
$$

The first term in (5.4) has the same form as Higgs contribution of the tensor product model if one takes, for instance, $w_1 = w_e + w_d$ and $w_2 = w_u$, but other combinations are possible. If $3w_d + 5w_e \neq 2$, the hybrid scalar model leads to electromagnetic interactions of neutral particles (Z-bosons, Higgs mesons). In terms of physical fields the contribution (4.21) takes the form

$$
\begin{aligned}
\mathcal{E}_{\rm HS} &= \frac{1}{2}\theta^{\mu\nu}\Bigg\{2\partial_\mu H\partial^\rho H + (H+v)^2\left(\frac{g^2}{2}W^+_\mu W^{-\rho} + \text{h.c.} + 2\mathcal{Z}_\mu\mathcal{Z}^\rho\right) \\
&\quad - \frac{1}{2}\delta^\rho_\mu\left[(\partial H)^2 + (H+v)^2\left(\frac{g^2}{2}W^+W^- + \mathcal{Z}^2\right) - \frac{\lambda}{8}H^2(H+2v)^2\right]\Bigg\} \\
&\quad \times\left(\mathrm{e}F_{\nu\rho} - 2\sin^2\theta\,\mathcal{Z}_{\nu\rho}\right) . \qquad (5.6)
\end{aligned}
$$

Electromagnetic interactions of neutral particles occur whenever hybrid maps become dynamical. In contrast to non-commutative covariant derivatives of tensor fields, the covariant derivative of a hybrid map destroys the relation (3.6) between gauge fields associated to left and right chiral fermions, already at first order in $\theta^{\mu\nu}$.

The Hybrid Fermion Model

The hybrid fermion model is based upon extending (3.24) to non-commutative space-time. This is achieved by constructing fermionic Seiberg–Witten maps transforming left and right, as right fermionic singlets and Higgs doublets, respectively. We decide to use a single Seiberg–Witten map for the Higgs doublet, although a version with two independent Higgs maps $\hat{\Phi}_1$, $\hat{\Phi}_2$ of φ and $\widetilde{\varphi}$ would be also conceivable. With the choice described in sect 3. the maps representing the Higgs fields are

$$
\begin{aligned}
\widehat{\Phi^*} &= \varphi^* + \frac{1}{2}\theta^{\mu\nu}\left[v^*_\mu\left(\partial_\nu + \frac{\mathrm{i}}{2}v^*_\nu\right) + a^* v^*_{\mu\nu}\right]\varphi^* , \qquad (5.7) \\
\hat{\Phi}^*_2 &= \widetilde{\varphi}^* + \frac{1}{2}\theta^{\mu\nu}\left[-\widetilde{v}^*_\mu\left(-\partial_\nu + \frac{\mathrm{i}}{2}\widetilde{v}^*_\nu\right) - a\widetilde{v}^*_{\mu\nu}\right]\widetilde{\varphi}^* . \qquad (5.8)
\end{aligned}
$$

Since (5.8) is obtained from $\hat{\Phi}_2$ by complex conjugation, it is constructed with the opposite Moyal product.

The action of the hybrid fermion model is given by

$$
\begin{aligned}
\mathcal{S}_{\rm HF} &= \int \mathrm{d}^4x\,\Bigg\{\bar{\hat{L}}^t_{\rm L}\star\mathrm{i}\left(\hat{\not{\mathcal{D}}}\star\hat{L}^t_{\rm L}\right) + w\bar{\hat{Q}}^t_{\rm L}\star\mathrm{i}\left(\hat{\not{\mathcal{D}}}\star\hat{Q}^t_{\rm L}\right) + w'\bar{\hat{Q}}'^t_{\rm L}\circ\mathrm{i}\left(\hat{\not{\mathcal{D}}}\circ\hat{Q}'^t_{\rm L}\right) \\
&\quad + \bar{\hat{e}}_{\rm R}\star\mathrm{i}\left(\hat{\not{\mathcal{D}}}\star\hat{e}_{\rm R}\right) + \bar{\hat{d}}_{\rm R}\star\mathrm{i}\left(\hat{\not{\mathcal{D}}}\star\hat{d}_{\rm R}\right) + \bar{\hat{u}}_{\rm R}\circ\mathrm{i}\left(\hat{\not{\mathcal{D}}}\circ\hat{u}_{\rm R}\right) \\
&\quad - \left(\bar{\hat{e}}_{\rm R}G_e\star\hat{L}^t_{\rm L}\star\widehat{\Phi^*} + \bar{\hat{d}}_{\rm R}G_d\star\hat{Q}^t_{\rm L}\star\widehat{\Phi^*} + \bar{\hat{u}}_{\rm R}G_u\circ\hat{Q}'^t_{\rm L}\circ\hat{\Phi}^*_2 + \text{h.c.}\right) \\
&\quad + \left(\hat{\mathcal{D}}^\rho\star\hat{\Phi}\right)^+\star\left(\hat{\mathcal{D}}_\rho\star\hat{\Phi}\right) + \frac{\lambda v^2}{4}\hat{\Phi}^+\star\hat{\Phi} - \frac{\lambda}{4}\hat{\Phi}^+\star\hat{\Phi}\star\hat{\Phi}^+\star\hat{\Phi} \\
&\quad - \frac{1}{2}{\sum_n}'\frac{1}{h_n^2}\,\mathrm{Tr}\,\hat{V}_{\rho\sigma}(n)\star\hat{V}^{\rho\sigma}(n) - \frac{1}{2h_u^2}\,\mathrm{Tr}\,\hat{V}_{\rho\sigma}(u)\circ\hat{V}^{\rho\sigma}(u)\Bigg\} . \qquad (5.9)
\end{aligned}
$$

Here $\hat{L}^t_{\rm L}$, $\hat{Q}^t_{\rm L}$, $\hat{L}'^t_{\rm L}$ are the hybrid Seiberg–Witten maps required by the gauge invariance of Yukawa couplings and the superscript t means transposition in electroweak isospin-hypercharge space. The quark maps are opposite to each other and occur with weights w, w', obeying $w + w' = 1$. Again we do not consider those contributions to the gauge sector which are associated to hybrid maps. Hence the primed sum goes over $n = e, d, \varphi$.

The consistency of the model can be examined in the same way as in previous models. As a consequence, the scaling parameters of the hybrid maps $\hat{\Psi}^t_{1\rm L}$, $\hat{\Psi}^t_{2\rm L}$ introduced in (3.25) are restricted to a single one

$$a'_{1\rm L} = -a_{1\rm L} = -a_{2\rm L} = a'_{2\rm L} \equiv a_{\rm L}\ . \tag{5.10}$$

After rescaling the fields, one can compute the non-commutative contribution of the various sectors of the Standard Model in terms of physical fields. Since both kind of maps (constructed with direct and opposite Moyal products) occur in the hybrid fermionic model, we expect to get electromagnetic and strong interactions which violate parity. This is apparent from the flavor preserving part contribution

$$\begin{aligned}
\Delta\mathcal{L}^{\rm FP}_{\rm HF} \;=\; & \sum_f \bar{f}\theta^{\mu\nu\rho}\boldsymbol{T_3}\Big\{{\rm i}\Big[-\Big(g_{\rm s}G_{\mu\nu} + \boldsymbol{Q}{\rm e}F_{\mu\nu} - 2\boldsymbol{Q}\sin^2\theta\,\mathcal{Z}_{\mu\nu}\Big)(\gamma_5 + 2\boldsymbol{w}P_{\rm L}) \\
& + 4(\boldsymbol{w}-1)\boldsymbol{T_3}P_{\rm L}{\rm e}F_{\mu\nu} + 2\left(1 + 2(\boldsymbol{w}-1)\cos(2\theta)\right)\boldsymbol{T_3}P_{\rm L}\mathcal{Z}_{\mu\nu}\Big] \\
& \times \Big[\nabla_\rho + 2{\rm i}\Big(\boldsymbol{T_3}P_{\rm L} - \boldsymbol{Q}\sin^2\theta\Big)\mathcal{Z}_\rho\Big] + \frac{g^2}{4}(1-2\boldsymbol{w})\Big(W^-_\mu W^+_{\nu\rho} + {\rm h.c.}\Big)P_{\rm L}\Big\} f \\
& -\frac{g^2}{4}W^+_\mu W^-_\nu \sum_f \bar{f}\theta^{\mu\nu\rho}(1-2\boldsymbol{w})\Big\{\overleftarrow{\nabla}_\rho + \nabla_\rho \\
& + 4{\rm i}\Big[\boldsymbol{Q}\sin^2\theta - \boldsymbol{T_3}\left(2 + \cos(2\theta)\right)\Big]\mathcal{Z}_\rho\Big\} P_{\rm L} f \\
& + (H+v)\sum_f \bar{f}\frac{1}{2}\theta^{\mu\nu}\frac{\boldsymbol{M}}{v}\Big\{2\boldsymbol{T_3}\Big[{\rm i}\overleftarrow{\nabla}_\mu\nabla_\nu + 2\Big(\boldsymbol{T_3}P_{\rm L} - \boldsymbol{Q}\sin^2\theta\Big)\overleftarrow{\nabla}_\nu\mathcal{Z}_\mu \\
& + 2\Big(\boldsymbol{T_3}P_{\rm R} - \boldsymbol{Q}\sin^2\theta\Big)\mathcal{Z}_\mu\nabla_\nu\Big] - \frac{1}{2}\Big(\mathcal{Z}_{\mu\nu} + {\rm i}g^2W^+_\mu W^-_\nu\Big)\Big\} f\ .
\end{aligned} \tag{5.11}$$

Even the lepton sector is changed in comparison with previous models. The electromagnetic interaction of the leptons is given by

$$\frac{1}{2}\theta^{\mu\nu}{\rm e}F_{\nu\rho}\left\{\frac{1}{2}\bar{e}\left[\delta^\rho_\mu\left({\rm i}\nabla\!\!\!\!/ - M_e\right) - 2{\rm i}\gamma^\rho\nabla_\mu\right]e + \bar{\nu}_{\rm L}\left(\delta^\rho_\mu{\rm i}\partial\!\!\!/ - 2{\rm i}\gamma^\rho\partial_\mu\right)\nu_{\rm L}\right\}\ . \tag{5.12}$$

The last term of (5.12) describes the photon-neutrino interaction considered in ref. [28]. Clearly, the hybrid map of left-chiral leptons $\hat{L}^t_{\rm L}$ is responsible for such a coupling.

For completeness we give in the following the contributions of the hybrid fermion model to the flavor changing part of the Standard Model:

$$\begin{aligned}
\Delta\mathcal{L}^{\rm l}_{\rm HF} \;=\; & -W^-_\rho\frac{g}{\sqrt{2}}\bar{e}\theta^{\mu\nu\rho}\left({\rm i}\overleftarrow{\nabla}_\mu\nabla_\nu - \overleftarrow{\nabla}_\mu\mathcal{Z}_\nu - \cos(2\theta)\mathcal{Z}_\mu\nabla_\nu - \frac{\rm e}{2}F_{\mu\nu} + \sin^2\theta\,\mathcal{Z}_{\mu\nu}\right)\nu_{\rm L} \\
& -(H+v)\frac{g}{\sqrt{2}}\bar{e}\frac{1}{2}\theta^{\mu\nu}\frac{M_e}{v}W^-_\mu\left(\overleftarrow{\nabla}_\nu - 2{\rm i}\mathcal{Z}_\nu\right)\nu_{\rm L}
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
\Delta\mathcal{L}^{\mathrm{q}}_{\mathrm{HF}} &= W^-_\rho \frac{g}{\sqrt{2}} \bar{d} V^+ \theta^{\mu\nu\rho} \left\{ -(w-w') \left[\mathrm{i} \overleftarrow{\nabla}_\mu \nabla_\nu + \left(\frac{4}{3}\sin^2\theta - 1\right) \overleftarrow{\nabla}_\mu \mathcal{Z}_\nu \right.\right. \\
&- \left.\left. \left(1 - \frac{2}{3}\sin^2\theta\right) \mathcal{Z}_\mu \nabla_\nu \right] + \frac{\mathrm{e}}{2} F_{\mu\nu} - \sin^2\theta \mathcal{Z}_{\mu\nu} \right\} P_{\mathrm{L}} u \\
&-(H+v) \frac{g}{\sqrt{2}} \bar{d} \frac{1}{2} \theta^{\mu\nu} \left\{ \frac{M_d}{v} V^+ P_{\mathrm{L}} W^-_\mu \left[\overleftarrow{\nabla}_\nu - 2\mathrm{i} \left(1 - \frac{2}{3}\sin^2\theta\right) \mathcal{Z}_\nu \right] \right. \\
&- \left. V^+ \frac{M_u}{v} P_{\mathrm{R}} \left[\nabla_\nu - 2\mathrm{i} \left(1 - \frac{1}{3}\sin^2\theta\right) \mathcal{Z}_\nu \right] W^-_\mu \right\} u \,. \qquad (5.14)
\end{aligned}$$

Finally, the contribution of the model to the gauge sector is

$$\Delta\mathcal{S}^{(g)}_{\mathrm{HF}} = \int \mathrm{d}^4x \left[\frac{1}{h_e^2} \Delta\mathcal{L}_e + \frac{1}{h_d^2} \Delta\mathcal{L}_d - \frac{1}{h_u^2} \Delta\mathcal{L}_u + \frac{1}{h_\varphi^2} \Delta\mathcal{L}_\varphi \right] . \qquad (5.15)$$

Notice that (5.15) contains only four weights $1/h_n^2$ restricted by the normalization conditions (3.11).

We expect qualitatively similar conclusions for the alternatively hybrid fermionic model with two independent non-commutative Higgs fields mentioned at the beginning of this subsection. In particular, the photon-neutrino interaction from (5.12) is the same.

6 Effective Low Energy Four-Fermi Interaction

In this section we will concentrate on the low energy aspects of the Standard Model in non-commutative space-time. As in the commutative Standard Model, energies below the masses of the weak bosons and Higgs (but larger than Λ_{QCD}) are considered low. Effective actions based upon Standard Model in non-commutative space-time were first considered in [20].

Here we obtain the effective action by integrating out the weak massive boson fields $W^\pm_\rho$, Z_ρ and H, while electromagnetic and strong interactions remain unintegrated. The corresponding free currents are

$$J^\pm_\rho = \frac{g}{\sqrt{2}} C^\pm_\rho \,, \qquad J^0_\rho = \frac{g}{\cos\theta} C^0_\rho \,, \qquad J_H = \frac{1}{v} C_H \qquad (6.1)$$

where

$$\begin{aligned}
C^-_\rho &\equiv \bar{e}\gamma_\rho \nu_{\mathrm{L}} + \bar{d} V^+ \gamma_\rho P_{\mathrm{L}} u \,, \qquad & C^0_\rho &\equiv \sum_f \bar{f}\gamma_\rho \left(\boldsymbol{T_3} P_{\mathrm{R}} - \boldsymbol{Q}\sin^2\theta\right) f \,, \\
C^+_\rho &\equiv \bar{\nu}_{\mathrm{L}} \gamma_\rho e + \bar{u} V \gamma_\rho P_{\mathrm{L}} d \,, \qquad & C_H &\equiv \sum_f \bar{f} \boldsymbol{M} f \,. \qquad (6.2)
\end{aligned}$$

Due to the factor $1/v$, only the top quark contribution to the Higgs free current C_H is of comparable order of magnitude as the other free currents. Moreover, the contributions of the Higgs path integral to the effective action are suppressed by a factor $1/m_H^2$ and will be ignored in the following.

The full weak currents are obtained by varying the extended action with respect to the fields $W_\rho^\pm$, Z_ρ and H. The part of the Lagrangian due to non-commutative space-time receives contributions from the usual three sectors

$$\Delta\mathcal{L} = \Delta\mathcal{L}^{\text{matter}} + \Delta\mathcal{L}^{\text{Higgs}} + \Delta\mathcal{L}^{\text{gauge}} \, . \tag{6.3}$$

Here $\Delta\mathcal{L}^{\text{matter}} = \Delta\mathcal{L}^{\text{l}} + \Delta\mathcal{L}^{\text{q}} + \text{h.c.} + \Delta\mathcal{L}^{\text{FP}}$, as given by (4.16) and (4.17), $\Delta\mathcal{L}^{\text{Higgs}}$ is a certain linear combination of (4.22) and (5.6) and $\Delta\mathcal{L}^{\text{gauge}}$ represents the contribution of the triple gauge boson interactions (see [18, 25]) relevant at low energies.

The construction of the effective action starts by computing the variation of $\Delta\mathcal{L}$ with respect to the massive bosonic field. Then, both fields and their variations are replaced by the convolution of free field propagators and the corresponding free currents. The effective action is obtained as a power series in inverse masses squared, but only the coefficients of lowest powers can be expressed in terms of Standard Model parameters. All the other coefficients depend in general on the regularization details.

It has been shown in Sect. 2 that the contribution of the fermions to the non-commutative effective action does not depend on the regularization details, because involving the totally antisymmetric tensor $\theta^{\mu\nu\rho}$. This property continues to be valid for the Standard Model extensions presented here. On the other hand, $\Delta\mathcal{L}^{\text{Higgs}}$ and $\Delta\mathcal{L}^{\text{gauge}}$ contain triple and quadruple boson interactions which are changed by renormalization. Moreover, such couplings in $\Delta\mathcal{L}^{\text{Higgs}}$ have positive mass dimension and can contribute to order $1/m^4$ in the effective action. Hence we will explicitly integrate only terms at most quadratically in the massive boson fields, thereby restricting our considerations to effective four-fermionic interactions.

The variation of $\Delta\mathcal{L}^{\text{matter}}$ is then given by

$$\delta\Delta\mathcal{L}_{\text{X}}^{\text{matter}} = \delta W_\rho^- \frac{g}{\sqrt{2}}\left[\left(\mathcal{C}_{\text{X}}^l\right)_+^\rho + (\mathcal{C}_{\text{X}}^q)_+^\rho\right] + \text{h.c.} + \delta Z_\rho \frac{g}{\cos\theta}\left(\mathcal{C}_{\text{X}}^{\text{FP}}\right)_0^\rho , \tag{6.4}$$

provided that the coefficient of δH is neglected. In (6.4) the subscript X stands for any of the four models discussed in previous sections. The coefficients of the field variations carry superscripts denoting the lepton (l), or quark (q) contributions to the flavor changing part, or of both, to the flavor preserving part (FP) for the non-commutative fermionic matter sector. The coefficients relevant to the effective four-fermionic interactions are given by

$$\left(\mathcal{C}_{\text{X}}^l\right)_+^\rho = \bar{e}\left[\theta^{\mu\nu\rho}\left(\text{i}\overleftarrow{\nabla}_\mu \nabla_\nu + \frac{\text{e}}{2}F_{\mu\nu}\right) - \frac{1}{2}\theta^{\rho\mu}M_e\nabla_\mu\right]\nu_{\text{L}} \, , \tag{6.5}$$

$$\left(\mathcal{C}_{\text{HF}}^l\right)_+^\rho = \bar{e}\left[\theta^{\mu\nu\rho}\left(-\text{i}\overleftarrow{\nabla}_\mu \nabla_\nu + \frac{\text{e}}{2}F_{\mu\nu}\right) + \frac{1}{2}\theta^{\rho\mu}M_e\nabla_\mu\right]\nu_{\text{L}} \, , \tag{6.6}$$

$$\begin{aligned}(\mathcal{C}_{\text{X}}^q)_+^\rho \quad &= \quad \bar{d}\Big[V^+\theta^{\mu\nu\rho}\left(\text{i}\overleftarrow{\nabla}_\mu \nabla_\nu - g_{\text{s}}G_{\mu\nu} - \frac{\text{e}}{6}F_{\mu\nu}\right)P_{\text{L}} \\ &\quad - \frac{1}{2}\theta^{\rho\mu}\left(M_d V^+ P_{\text{L}}\nabla_\mu + V^+ M_u P_{\text{R}}\overleftarrow{\nabla}_\mu\right)\Big]\, u,\end{aligned} \tag{6.7}$$

$$
\begin{aligned}
(\mathcal{C}^q_{\rm TM})^\rho_+ &= \bar d\Big[(w-w')V^+\theta^{\mu\nu\rho}\Big(\mathrm{i}\overleftarrow{\nabla}_\mu\nabla_\nu - g_{\rm s}G_{\mu\nu} - \frac{\mathrm{e}}{6}F_{\mu\nu}\Big)P_{\rm L} \\
&\quad - \frac{1}{2}\theta^{\rho\mu}\Big(M_dV^+P_{\rm L}\nabla_\mu - V^+M_uP_{\rm R}\overleftarrow{\nabla}_\mu\Big)\Big]u, \qquad (6.8)
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{C}^q_{\rm HF})^\rho_+ &= \bar d\Big\{\theta^{\mu\nu\rho}V^+\Big[-(w-w')\mathrm{i}\overleftarrow{\nabla}_\mu\nabla_\nu + \frac{\mathrm{e}}{2}F_{\mu\nu}\Big]P_{\rm L} \\
&\quad + \frac{1}{2}\theta^{\rho\mu}\Big(M_dV^+P_{\rm L}\nabla_\mu - V^+M_uP_{\rm R}\overleftarrow{\nabla}_\mu\Big)\Big\}u\ , \qquad (6.9)
\end{aligned}
$$

$$
\begin{aligned}
\left(\mathcal{C}^{\rm FP}_{\rm X}\right)^\rho_0 &= \sum_f \bar f\Big\{\theta^{\mu\nu\rho}\Big(\boldsymbol{T_3}P_{\rm L} - \boldsymbol{Q}\sin^2\theta\Big)\Big(\mathrm{i}\overleftarrow{\nabla}_\mu\nabla_\nu - g_{\rm s}G_{\mu\nu} - \boldsymbol{Q}\mathrm{e}F_{\mu\nu}\Big) \qquad (6.10) \\
&\quad - \frac{1}{2}\theta^{\rho\mu}\boldsymbol{M}\Big[\Big(\boldsymbol{T_3}P_{\rm L} - \boldsymbol{Q}\sin^2\theta\Big)\nabla_\mu + \Big(\boldsymbol{T_3}P_{\rm R} - \boldsymbol{Q}\sin^2\theta\Big)\overleftarrow{\nabla}_\mu\Big]\Big\}f\ ,
\end{aligned}
$$

$$
\begin{aligned}
\left(\mathcal{C}^{\rm FP}_{\rm TM}\right)^\rho_0 &= 2\sum_f \bar f\boldsymbol{T_3}\Big\{-\theta^{\mu\nu\rho}\left(\gamma_5 + 2\boldsymbol{w}P_{\rm L}\right)\Big(\boldsymbol{T_3}P_{\rm L} - \boldsymbol{Q}\sin^2\theta\Big) \\
&\quad \times\Big(\mathrm{i}\overleftarrow{\nabla}_\mu\nabla_\nu - g_{\rm s}G_{\mu\nu} - \boldsymbol{Q}\mathrm{e}F_{\mu\nu}\Big) \qquad (6.11) \\
&\quad + \frac{1}{2}\theta^{\rho\mu}\boldsymbol{M}\Big[\Big(\boldsymbol{T_3}P_{\rm L} - \boldsymbol{Q}\sin^2\theta\Big)\nabla_\mu + \Big(\boldsymbol{T_3}P_{\rm R} - \boldsymbol{Q}\sin^2\theta\Big)\overleftarrow{\nabla}_\mu\Big]\Big\}f\ ,
\end{aligned}
$$

$$
\begin{aligned}
\left(\mathcal{C}^{\rm FP}_{\rm HF}\right)^\rho_0 &= 2\sum_f \bar f\boldsymbol{T_3}\Big\{\theta^{\mu\nu\rho}\Big[\Big(\left(1 + 2(\boldsymbol{w}-1)\cos(2\theta)\right)\boldsymbol{T_3}P_{\rm L} + 2\sin^2\theta\left(\gamma_5 + 2\boldsymbol{w}P_{\rm L}\right)\boldsymbol{Q}\Big) \\
&\quad \times\Big(\mathrm{i}\overleftarrow{\nabla}_\mu\nabla_\nu - g_{\rm s}G_{\mu\nu} - \Big(\boldsymbol{Q} - \frac{1}{\sin^2\theta}\boldsymbol{T_3}P_{\rm L}\Big)\mathrm{e}F_{\mu\nu}\Big) \qquad (6.12) \\
&\quad + \Big(1 + (\boldsymbol{w}-1)\cos^2\theta\Big)\Big(\boldsymbol{T_3}g_{\rm s}G_{\mu\nu} - \frac{1}{4\sin^2\theta}\mathrm{e}F_{\mu\nu}\Big)P_{\rm L}\Big] \\
&\quad - \frac{1}{2}\theta^{\rho\mu}\boldsymbol{M}\Big[\Big(\boldsymbol{T_3}P_{\rm L} + \boldsymbol{Q}\sin^2\theta\Big)\nabla_\mu + \Big(\boldsymbol{T_3}P_{\rm R} + \boldsymbol{Q}\sin^2\theta\Big)\overleftarrow{\nabla}_\mu\Big]\Big\}f\ .
\end{aligned}
$$

Except for (6.5) which is valid for $\mathrm{X} = \mathrm{PM}, \mathrm{TM}$ and HS, the subscript X refers to both direct tensor product (PM) and hybrid scalar (HS) models.

Let us discuss now the contribution of the Higgs sector to the effective action. A look at (4.22) shows that it contains triple and quadruple boson interactions proportional to v^2, i.e to mass squared. The only contribution quadratic in the massive vector bosons is coming from the electromagnetic interaction

$$
\mathrm{e}m_W^2\frac{1}{2}\theta^{\mu\nu}F_{\nu\rho}\Big(W^+_\mu W^{-\rho} + \text{h.c.} - \frac{1}{2}\delta^\rho_\mu W^+W^-\Big)\ . \qquad (6.13)
$$

In the hybrid scalar model there is, as follows from (5.4) and (5.6) an additional electromagnetic interaction contributing to the effective action

$$\begin{aligned}&\mathrm{e}\frac{1}{2}\theta^{\mu\nu}F_{\nu\rho}\Big\{2\partial_\mu H\partial^\rho H-\frac{1}{2}\delta^\rho_\mu\Big[(\partial H)^2-\frac{m_H^2}{2}H^2\Big]\\&+m_W^2\Big[2\Big(W^+_\mu W^{-\rho}+\text{h.c.}+\frac{1}{\cos^2\theta}Z_\mu Z^\rho\Big)-\delta^\rho_\mu\Big(W^+W^-+\frac{1}{2\cos^2\theta}Z^2\Big)\Big]\Big\}\end{aligned}\qquad(6.14)$$

The last two terms in (6.14) give a contribution of order $1/m^4$ in the effective action and can be neglected.

For the same reason one can neglect the contribution of $\Delta\mathcal{L}^{\text{gauge}}$.

According to (A.4) the effective action is obtained from (6.4) and from $\Delta\mathcal{L}^{\text{Higgs}}$ by means of the following replacements:

$$\delta W^\pm_\rho, W^\pm_\rho \Rightarrow -\frac{g}{m_W^2\sqrt{2}}C^\pm_\rho\ ,\qquad \delta Z_\rho, Z_\rho\Rightarrow -\frac{g}{m_Z^2\cos\theta}C^0_\rho\ .\qquad(6.15)$$

The effective Hamiltonian is given by

$$\begin{aligned}\Delta\mathcal{H}^{\text{eff}}_{\text{X}} \ =\ & \frac{4G_F}{\sqrt{2}}\Big\{C^-_\rho\Big[\big(\mathcal{C}^l_{\text{X}}\big)^\rho_+ + (\mathcal{C}^q_{\text{X}})^\rho_+\Big]+\text{h.c.}+2C^0_\rho\big(\mathcal{C}^{\text{FP}}_{\text{X}}\big)^\rho_0\\&-\frac{1}{2}\theta^{\mu\nu}\mathrm{e}F_{\nu\rho}\Big[(\kappa_{\text{X}}+2\mu_{\text{X}})\Big(C^+_\mu C^{-\rho}+\text{h.c.}-\delta^\rho_\mu\frac{\kappa_{\text{X}}}{2}C^+\cdot C^-\Big)\\&+\mu_{\text{X}}\Big(4C^0_\mu C^{0\rho}-\delta^\rho_\mu(C^0)^2\Big)\Big]\Big\}\end{aligned}\qquad(6.16)$$

where $G_F/\sqrt{2}\equiv g^2/(8m_W^2)$ is the Fermi weak constant and

$$\begin{aligned}&\kappa_{\text{PM}}=w_1-w_2\ ,\qquad \kappa_{\text{TM}}=\kappa_{\text{HF}}=1\ ,\qquad \kappa_{\text{HS}}=w_e+w_d-w_u=1-2w_u\ ,\\&\mu_{\text{PM}}=\mu_{\text{TM}}=\mu_{\text{HF}}=0\ ,\qquad \mu_{\text{HS}}=\frac{2w_u-w_d}{3}-w_e\ .\end{aligned}\qquad(6.17)$$

Eq. (6.16) summarizes the non-commutative contribution of the various models to the processes involving four fermions. Since $U_{\text{em}}\otimes SU_{\text{c}}$ gauge symmetry is explicitly preserved, the formula describes also processes with one or more photons or / and gluons. In computing the corresponding transition probability amplitudes one must consider, besides the point interaction (6.16) tree diagrams in which one of the fermionic lines is off-shell.

7 Conclusions

The implementation of Yukawa interactions in non-commutative space-time provided several options for extending the Standard Model, all of them being based upon the gauge group $U_{\text{Y}}(1)\otimes SU_{\text{L}}(2)\otimes SU_{\text{c}}(3)$. Each extension was obtained by assigning certain Seiberg–Witten maps to the fields entering the commutative Yukawa interaction. In tensor product models, one of the maps was transforming as the tensor product of the other two. Alternatively, the map associated to the central factor in the Yukawa product transformed left and right, as the maps corresponding to left- and right-handed fields, respectively. In

this way we obtained hybrid non-commutative models. A further proliferation of models appeared because the couplings to Higgs and to its charge conjugate field were independent of each other. The complete non-commutative extension could be achieved with the same star product, or through opposite (complex conjugate) Moyal products. In case that inequivalent Seiberg–Witten maps were associated to the same commutative matter representation, we assumed a weighted contribution to the non-commutative action.

In particular, we derived, at lowest order in $\theta^{\mu\nu}$, the non-minimal Non-Commutative Standard Model within the class of tensor products models. Also, by making hybrid Seiberg–Witten maps dynamical, we found that the corresponding non-commutative models always predict an electromagnetic coupling of neutral particles, like Z-boson, Higgs meson or neutrino.

We evaluated also low energy effective actions for the non-commutative models, by integrating out massive bosonic degrees of freedom. If pure fermionic matter coupled to massive vector bosons of mass m, we derived an effective action for four- and six-fermion processes, valid up to order $1/m^4$ included. On the other hand, due to the Higgs mechanism, non-commutative extensions of the Standard Model contain, in general, triple and quadruple boson couplings of positive mass dimension. Such couplings being renormalized were discarded, reducing the validity of the effective actions to order $1/m^2$. We obtained formulas for the low energy effective interaction Hamiltonian of four fermions and a number of photons and gluons limited only by the $U_{\mathrm{em}}(1) \otimes SU_{\mathrm{c}}(3)$ gauge symmetry.

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A Appendix

We shall give here a formula for the effective action obtained by integrating out massive bosonic fields coupled to currents. Our starting point is the action

$$\frac{1}{2}\xi_i K_{ij}\xi_j + u_i\xi_i - V(\xi)\ . \tag{A.1}$$

In (A.1) we use a highly compact notation in which space time coordinates and field indices are included in Latin indices $i, j, k, \ldots$, summation occurs whenever an index is repeated. The basic fields (upon one will integrate) are denoted by ξ_i and the (free) currents by u_i, the symmetric matrix K_{ij} in the kinetic term is nonsingular, its inverse being the free propagator. We also assume $V(\xi)$ to be polynomial of some degree N in the field ξ and derivatives

$$V(\xi) = \sum_{\alpha=1}^{N} \frac{1}{\alpha} C_{i_1\ldots i_\alpha}\xi_{i_1}\cdots\xi_{i_\alpha}\ . \tag{A.2}$$

The effective action $\mathcal{S}^{\rm eff}(u)$ is defined to be the part of the path integral

$$\frac{1}{\mathrm{i}}\ln\int\left(\prod_k \mathrm{d}\xi_k\right)\exp\left\{\mathrm{i}\left[\frac{1}{2}\xi_i K_{ij}\xi_j + u_i\xi_i - V(\xi)\right]\right\} \tag{A.3}$$

linear in the coefficients $C_{i_1\ldots i_\alpha}$.

One obtains

$$\mathcal{S}^{\rm eff}(u) = -\frac{1}{2}u_i\bar{u}_i + 2\bar{u}_i J_i(0) - \int_0^1 \mathrm{d}a\, \bar{u}_i J_i(a\bar{u}) - \mathcal{P}_{N-2}(\bar{u}) \tag{A.4}$$

where $\bar{u}_i \equiv K^{-1}_{ij}u_j$ and $J_i(\xi)$ is the full current defined by $\delta V = J_i(\xi)\delta\xi_i$. In terms of the coefficients introduced in (A.2) the full current has the following expansion:

$$J_i(\xi) = C_i + C_{ij}\xi_j + C_{ijk}\xi_j\xi_k + \cdots\ . \tag{A.5}$$

Finally, $\mathcal{P}_{N-2}(\bar{u})$ is a certain polynomial of degree $N-2$ in $\bar{u}$. For the purpose of this paper the case $N=6$, considered below, is sufficient. We have

$$\begin{aligned}
\mathcal{P}_4(\bar{u}) \ = \ & C_{ijk}\frac{K^{-1}_{ij}}{\mathrm{i}}\bar{u}_k + \frac{3}{4}C_{ijkl}\left(\frac{K^{-1}_{ij}}{\mathrm{i}}\frac{K^{-1}_{kl}}{\mathrm{i}} + 2\frac{K^{-1}_{ij}}{\mathrm{i}}\bar{u}_k\bar{u}_l\right)\\
& + C_{ijklm}\left(3\frac{K^{-1}_{ij}}{\mathrm{i}}\frac{K^{-1}_{kl}}{\mathrm{i}}\bar{u}_m + 2\frac{K^{-1}_{ij}}{\mathrm{i}}\bar{u}_k\bar{u}_l\bar{u}_m\right)\\
& + \frac{5}{2}C_{ijklmn}\left(\frac{K^{-1}_{ij}}{\mathrm{i}}\frac{K^{-1}_{kl}}{\mathrm{i}}\frac{K^{-1}_{mn}}{\mathrm{i}} + 3\frac{K^{-1}_{ij}}{\mathrm{i}}\frac{K^{-1}_{kl}}{\mathrm{i}}\bar{u}_m\bar{u}_n + \frac{K^{-1}_{ij}}{\mathrm{i}}\bar{u}_k\bar{u}_l\bar{u}_m\bar{u}_n\right)\ .
\end{aligned} \tag{A.6}$$

In a local field theory the factors K^{-1}_{ij} in (A.7) represent free propagators at zero distance and are divergent quantities. Without entering the regularization details one can estimate the importance of the various terms in $\mathcal{P}_4(\bar{u})$ by simple dimensional analysis.

Let us apply (A.4) to get the effective action in non-commutative space-time for the simple theory described in Sect. 2, where the free currents u_i are given by $\bar{\psi}\gamma^\rho t_a\psi$. Due

to the antisymmetry of $\theta^{\mu\nu\rho}$, the matter sector represented by (2.14) does not yield any contribution to the corresponding polynomial. Contributions are however expected from the kinetic term for non-commutative vector fields. Notice that each factor $\bar{u}_i$, or K^{-1}_{ij} introduces an inverse power of mass squared. Referring to (2.13) one easily finds that the contribution of (A.7) to the effective four-fermion interaction is given by

$$\frac{3}{2}\left(C_{ijkl}\frac{K^{-1}_{ij}}{\mathrm{i}}\bar{u}_k\bar{u}_l + 5C_{ijklmn}\frac{K^{-1}_{ij}}{\mathrm{i}}\frac{K^{-1}_{kl}}{\mathrm{i}}\bar{u}_m\bar{u}_n\right) \ . \tag{A.7}$$

Since both C_{ijkl} and C_{ijklmn} do not introduce factors of order m, the contribution of (A.7) is of order $1/m^6$.

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